ON SOME COMBINATORIAL ASPECTS OF
REPRESENTATION THEORY

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A dissertation submitted to the
Graduate School—New Brunswick
Rutgers, The State University of New Jersey
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy
Graduate Program in Mathematics

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New Brunswick, New Jersey
May, 2004
We study three different but related problems: The computational complexity of character formulas, recursion formulas for characters and weight multiplicities of representations of Lie groups and Lie algebras, and the Pieri formula for nonsymmetric Jack polynomials. In the first work, we show that, with respect to counting the total number of arithmetic operations, a relatively new formula of Sahi for computing characters of representations of Lie groups and Lie algebras is optimal while the classical formula of Freudenthal is not. In the second work, we give new recursions for computing characters and weight multiplicities of Lie algebras which generalize a well known recursion for type \( A_2 \). In the third and final work, we give a complete answer for the degree one Pieri rule for the product of two nonsymmetric Jack polynomials.
Acknowledgements

I own immense debt to my supervisor, Dr. Siddhartha Sahi, for his sound advice and careful guidance in all stages of my research. To all who directly or indirectly cooperated and supported me in accomplishing the present work, including, but not limited to, fellow graduate students and friends Klay Kruczek, Aaron Lauve, Chris Long, David Nacin, Francisco Ojeda, Alfredo Rios, Eric Sundberg and Liangyi Zhao, and to my friends Michelle, Diwakar, Letícia, Diogo, Emília, and Hélio, I would like to extend my deepest appreciation. Also, I would like to thank Professors Roe Goodman and James Abello for their suggestions and for their careful proof-reading of an early version of this dissertation. Last, but not least, I would like to thank Professors Paulo A. S. Caetano and João Sampaio, for their encouraging support, and Prof. Sadao Massago, who generously offered to take on my teaching load during the few weeks preceding my defense. I was partially supported by CAPES during the first 4 years of the Ph.D. program.
Dedication

For my dedicated wife, whose love, patience and caring support helped me persevere throughout this journey. For my mother, who resigned so many things for me to have a broader education.

À minha dedicada esposa, por seu amor, paciência e apoio que me ajudaram a ser perseverante ao longo desta jornada. À minha mãe, que com resignação aceitou deixar de lado muitas coisas para que eu pudesse ter uma educação mais abrangente.
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Chapter 1

A new algorithm for computing characters of simple Lie algebras

This chapter deals with a new algorithm for computing characters of representations of a simple Lie algebra. We will introduce and compare this algorithm with the well-known method of Freudenthal and show that, for the task of computing a whole character table, the new algorithm will perform better than Freudenthal’s and will actually be of optimal order.

1.1 Introduction

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra of rank $n$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $P \subset \mathfrak{h}^*$ the weight lattice of $\mathfrak{g}$ relative to $\mathfrak{h}$. For $\lambda$ in $P$ let $\pi^\lambda : \mathfrak{g} \to \text{End}(V^\lambda)$ be the finite dimensional irreducible representation of $\mathfrak{g}$ of highest weight $\lambda$. The character of $\pi^\lambda$ relative to $\mathfrak{h}$ is defined by

$$\chi^\lambda(H) = \text{Tr}(\exp \pi^\lambda(H)),$$

for all $H$ in $\mathfrak{h}$. If $P^\lambda \subset P$ is the weight system of $\pi^\lambda$ then we can expand the character as the sum

$$\chi^\lambda(H) = \sum_{\mu \in P^\lambda} m^\lambda(\mu)e^{\mu(H)},$$

Here the coefficients $m^\lambda(\mu)$, called weight multiplicities, are the dimensions of the weight spaces

$$V(\mu) = \{v \in V \mid \pi^\lambda(H)v = \mu(H)v \text{ for all } H \text{ in } \mathfrak{h}\}$$

of $\pi^\lambda$ relative to $\mathfrak{h}$. When $\mu$ is not a weight of $\pi^\lambda$ then we define $m^\lambda(\mu)$ equal to zero.

In many practical as well as theoretical situations two interesting questions arise:
1. What are the weights in $P^\lambda$?

2. What are the multiplicities $m_\lambda(\mu)$?

The second question is harder since it implies an answer to the first, so it seems natural to try to answer them in direct order. This is the case when one uses for instance Freudenthal’s formula (1.3). However, some methods like the one we present here work by finding the character directly, which provides answer to both questions simultaneously.

A few methods for computing multiplicities and characters have been around for some time now. It seems that the most popular ones are based on some variation of Freudenthal’s formula (1.3), expressing each multiplicity as an integral linear combination involving only the multiplicities of “higher” weights.

Some methods based on Kostant’s and Racah’s multiplicity formulas have also been investigated, but these don’t seem to be of widespread use in practice, perhaps due to the fact that they require summing over the entire Weyl group.

More recently, Littlemann introduced a very elegant combinatorial method in which the multiplicities are obtained by counting the number of paths in the weight lattice connecting a weight with the highest weight. However, it seems that the practical implementation of this method is still in course.

The main goal of this chapter is to introduce a new algorithm (Algorithm 2) for computing the character, study its complexity and compare it with that of a straight implementation of Freudenthal’s formula (Algorithm 1). For this purpose, we use [1], [12] and [23] as references to complexity analysis. In particular, we assume the straight-line program model with uniform cost function [1, §1.5(I)], so that the time complexity is determined by counting the number of arithmetic operations.

This new algorithm is based on the discovery of a recursion and a new character formula (Theorem 1.5.1) by Sahi [35], following developments by Cherednik [8], Heckman [13] and Opdam [33]. The remarkable fact that we show is that in a certain sense the new algorithm is optimal.

The main task we want to consider is that of computing the character table $T_m$ for a fixed Lie algebra $\mathfrak{g}$ up to some given “depth” $m$. For our purposes we will define depth
the same as the euclidean length, but occasionally will be using another convenient notion of length coming from the Weyl group. Hence we define $T_m = \{ m_\lambda(\mu) \mid \lambda \in P^+, \|\lambda\| \leq m \text{ and } \mu \in P_\lambda \}$, the character table of $\mathfrak{g}$ of depth $m$.

Recall that $P_\lambda$ is the set of weights that occur in $\pi_\lambda$ with positive multiplicity, $P^+$ is the cone of dominant weights in $P$ and $P_m^+ = \{ \mu \in P^+ \mid \|\mu\| \leq m \}$ is the set of dominant weights $\mu$ with length smaller than or equal to $m$. In what follows, we shall need the following estimates for the sizes of these sets when the rank $n$ is fixed.

**Proposition 1.1.1** For a dominant integral weight $\lambda$ of length $m = \|\lambda\|$ we have the following estimates:

1. $|P_\lambda| = O(m^n)$,
2. $|P_m^+| = O(m^n)$,
3. $|T_m| = O(m^{2n})$.

As a direct consequence of these estimates we obtain the complexity of the best possible algorithm for computing $T_m$.

**Corollary 1.1.2** An algorithm for computing the character table up to depth $m = \|\lambda\|$ has time complexity at least $O(m^{2n})$.

Paraphrasing what we said above, there are mainly two approaches to the problem of computing multiplicities:

1. Computing each individual multiplicity $m_\lambda(\mu)$ separately using, say, a recursion formula like Freudenthal’s.
2. Finding the character $\chi_\lambda$ directly and reading the multiplicities $m_\lambda(\mu)$ off from its coefficients.

Both have advantages and disadvantages as it can be seen from the following results:

**Theorem 1.1.3 (Complexity of Freudenthal’s Algorithm)** If $\lambda$ is a dominant integral weight of length $m$, then Algorithm 1 computes a single multiplicity in $O(m)$ time.
Therefore, it can be used to compute the character \( \chi_\lambda \) in \( O(m^{n+1}) \) time and the full character table \( T_m \) in \( O(m^{2n+1}) \) time.

**Theorem 1.1.4 (Complexity of the new algorithm)** If the weight \( \lambda \) is dominant integral of length \( m \), Algorithm 2 computes the character \( \chi_\lambda \) in \( O(m^{n+1}) \) time. However, it computes the full character table \( T_m \) in optimal \( O(m^{2n}) \) time.

An important remark is that the bounds we obtained are actually lower bounds since it will become clear that there is room for improvement only in the constants involved.

On the other hand, these constants can be seen to depend exponentially on \( n \) if we let the rank \( n \) vary. Even with powerful computers available nowadays, it seems to be quite challenging to investigate representations of Lie algebras of ranks higher than 10, or even representations of the exceptional Lie algebra \( E_8 \).

Here is an outline of this chapter: after notational considerations and a brief summary of the relevant results of Sahi, Opdam and Heckman, we study the complexity of Freudenthal’s algorithm. Then we introduce and analyze the complexity of the new algorithm for computing the character. In these sections we prove our main results, 1.1.1-1.1.4 above. In the final section we give a summary of the computational experience we acquired. The remainder of this introduction is motivational.

It is well-known that representation theory of Lie algebras and Lie groups have applications in and connections with many branches in Mathematics and Mathematical Physics. It is often important in these areas to find explicit realizations of representations and their characters. As an example of this interaction, characters of Lie algebras of type \( A \) correspond to Schur functions in Algebraic Combinatorics, and representation-theoretic properties of the former give rise to interesting combinatorial properties of the latter and vice-versa.

It was found relatively recently that the representation theory of certain Hecke algebras play a key role in the proof of some interesting facts about the so-called Jack polynomials [18], their immediate generalization, called Jacobi polynomials [33], and their \( q \)-analogues, called Macdonald polynomials [28]. This lead to important developments in the areas of Special Functions, Algebraic Combinatorics, and Representation Theory (cf. [19] or
One of such developments, namely the study of certain modules for graded Hecke algebras, has lead to the discovery of Sahi’s character formula. Even more surprising, a specialization of the same formula gives the spherical function of symmetric spaces. In this sense, Sahi’s formula is presently the most explicit way of describing these functions.

Nowadays with the advent of powerful, inexpensive computers it has become practical to routinely carry out calculations involving character formulas and weight multiplicities. Following the pioneering work of Krusemeyer [26], who wrote one of the first implementations of Freudenthal’s formula using ALGOL 60, Moody & Patera [30] developed and used an efficient computer implementation of that same formula to write down the first reasonably large character tables for all classical and exceptional Lie groups. Beginning in 1988, Leeuwen, Cohen and Lisser [40] have developed LiE, a comprehensive computer algebra system for Lie-theoretic calculations, which uses Moody & Patera version of Freudenthal’s algorithm [9, §8.6] [24] [40] [30]. The source code for this system is available to the public and can be downloaded from its authors’ website.

The practicability of computing with certain character formulas has been addressed before in a work by Kolman & Beck [24]. In their survey, they study implementations of formulas by Freudenthal, Kostant and Racah, however their approach to performance analysis is mainly based on experimental data. In another survey, Stembridge [38] presents what he believes are some of the most interesting computational challenges in Lie Theory at the moment. As he points out, many of the techniques that work well for groups of type $A$ happen to fail dramatically for groups like $E_8$, mainly because of the huge size of its Weyl group. He states that based on his own experience, a good measure for the “complexity” of an irreducible representation $V^\lambda$ is the size of $P^+ \cap P^\lambda$, i.e. the number of distinct orbits of weights in $P^\lambda$ under the Weyl group.

From this we can see that although the interest and the need for better algorithms in Lie Theory have been around for some time now, it seems that a systematic study of the computational complexity of character formulas has never been addressed before. What we present in this chapter is a first step in that direction.
1.2 Notation

The purpose of this section is to fix the representation-theoretic notation. As in the introduction, let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ and $R_0$ the corresponding root system. Fix $\Delta = \{\alpha_1, \ldots, \alpha_n\}$, a choice of simple roots in $R_0$, and let $R_0^+$ be the corresponding positive roots. Also let $E_0$, $(\ , \ )$ be the $n$-dimensional real euclidean space spanned by $\Delta$ and identify its complexification with $\mathfrak{h}^*$. We say that $R_0$ is a root system of rank $n$. The $\mathbb{Z}$-span of $R_0$ in $E_0$ is the root lattice $Q$ of $\mathfrak{g}$. As usual, we define the coroots $\alpha^\vee$ as $2\alpha/(\alpha, \alpha)$ for all $\alpha$ in $R_0$. The (finite) Weyl group $W_0$ of $R_0$ is the subgroup of $O(n, \mathbb{R})$ which stabilizes $R_0$. It is the Coxeter group with generators $s_1, \ldots, s_n$, where $s_i \cdot \lambda = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$, for all $\lambda$ in $E_0$, are called simple reflections. Each $s_i$ fixes pointwise the hyperplane $H_{\alpha_i,0} = \{x \in E_0 | (\alpha_i, x) = 0\}$, exchanges $\alpha_i$ with $-\alpha_i$, and permutes all other positive roots. We use the partial ordering on $E_0$ given by $\lambda \geq \mu$ if and only if $\lambda - \mu$ is a non-negative sum of positive roots. With respect to this ordering, $R_0$ has a unique highest (long) root $c_1\alpha_1 + \cdots + c_n\alpha_n$ and a unique highest short root $\beta$.

At this point, since this number appears throughout, it is interesting to note that the order of the Weyl group $W_0$ can be computed directly from the information encoded in the root system $R_0$, namely

$$|W_0| = n!c_1 \cdots c_n d,$$

where $d$ is the determinant of the Cartan matrix $((\alpha_i, \alpha_j^\vee))_{i,j=1..n}$.

If $\pi : \mathfrak{g} \to \text{End}(V)$ is a representation of $\mathfrak{g}$, then $V$ can be seen as a $\mathfrak{g}$-module with the Lie algebra action defined as $x \cdot v = \pi(x)(v)$. We define the weight subspaces of $V$ relative to $\mathfrak{h}$ as

$$V(\mu) = \{v \in V | h \cdot v = \mu(h)v, \text{ for all } h \in \mathfrak{h}\}.$$ 

The elements $\mu$ in $\mathfrak{h}^*$ for which $V(\mu)$ is non-zero are called weights of $V$, and the multiplicity $m(\mu)$ of a weight $\mu$ is just $\dim V(\mu)$. Whenever $(\mu, \alpha^\vee) \in \mathbb{Z}$ for all simple roots $\alpha$, we say that $\mu$ is an integral weight. If all these numbers are non-negative we say that $\mu$ is dominant integral. When $V$ is irreducible and finite dimensional, then $V$ is a
(finite) direct sum of its weight spaces, all the weights in $V$ are integral, there is a unique
dominant integral highest weight $\lambda$ and the corresponding highest weight space $V(\lambda)$ is
one-dimensional. We denote such a representation by $V^\lambda$, and the corresponding weights
that occur by $P^\lambda$. All weights $\mu$ in $P^\lambda$ also have the property that $(\mu, \alpha^\vee) \in \mathbb{Z}$ for all
$\alpha \in R_0$, and are thus called integral weights. The subset $P$ of all integral weights in $E_0$,
called the weight lattice of $\mathfrak{g}$, is a lattice containing the root lattice $Q$ as a sublattice of
index $d$. The weights $\mu$ in $P$ which satisfy $(\mu, \alpha^\vee) \geq 0$ for all $\alpha \in R_0^+$ are called dominant,
and the set of all dominant integral weights $P^+$ forms a convex cone in $P$.

A key role in Sahi’s formula is played by the dual affine root system $R$ and its associated
dual affine Weyl group $W$. Let $E$ be the space of affine linear functions on $E_0$. This can
be identified with the vector space $E = \mathbb{R}\delta \oplus E_0$ via the pairing $(r\delta + x, y) = r + (x, y)$,
for $r \in \mathbb{R}$ and $x, y \in E_0$. The dual affine root system $R$ is the subset of $E$ given by
$\{m\delta + \alpha^\vee | m \in \mathbb{Z}, \alpha \in R_0\}$. If $\beta$ is the highest short root of $R_0$, then a base for $R$ is given
by $a_0 = \delta - \beta^\vee$, $a_1 = \alpha_1^\vee$, $\ldots$, $a_n = \alpha_n^\vee$, and we write $s_0$ for the affine reflection through
the affine hyperplane $H_{a_0,0} = \{x \in E | (a_0, x) = 0\} = \{x \in E | (\beta^\vee, x) = 1\} = H_{\beta^\vee,1}$. The
dual affine Weyl group $W$ is the Coxeter group generated by the affine reflection $s_0$ along
with $s_1, \ldots, s_n$, so that the finite Weyl group $W_0$ is a subgroup of $W$, and $W$ acts on the
weight lattice $P$. The fundamental domain for this action is the set

$$\mathcal{O} = \{\lambda \in P | (\alpha^\vee, \lambda) = 0 \text{ or } 1, \forall \alpha \in R_0^+\},$$

which contains exactly one (minuscule) weight for each orbit of $W$. The cardinality of
this set is precisely $d$, hence it contains only the zero weight for $\mathfrak{g}$ of types $E_8$, $F_4$ and
$G_2$, contains the zero weight along with all the $n$ fundamental weights for type $A_n$, or
contains the zero weight and 2-4 fundamental weights for the remaining types.

The length of an element $w$ in $W$ is defined as the smallest $m$ for which $w = s_{i_1} \cdots s_{i_m}$,
where $s_{i_j}$ is one of the simple reflections $s_0, s_1, \ldots, s_n$. An expression for $w$ of this form
is called reduced. The action of $W$ on $E$ causes this space to be partitioned by (infinitely
many) hyperplanes. The connected components of the complement to the hyperplanes
are called alcoves and the collection of all alcoves is denoted $\mathcal{A}$. It can be shown that
although the action of $W$ on $E$ is not transitive in general, it is simply transitive on $\mathcal{A}$. 
This allows us to define the length of a weight $\lambda \in P$ as the length of the unique shortest element $w_\lambda$ of $W$ which takes $\lambda$ to $\overline{\lambda} := w_\lambda(\lambda) \in \mathcal{O}$. Geometrically this is the same as the number of hyperplanes separating the alcove of $\lambda$ from $\mathcal{O}$ [16, §4.4-4.5]. This allows us to compare this group theoretic length with the euclidean length of a vector in one of the alcoves. Since the hyperplanes are evenly spaced, these lengths are asymptotically proportional. In terms of the complexity analysis, this proportionality is absorbed by the constants involved, which allows us to use both notions of length interchangeably.

1.3 Proof of the estimates

For a given finite dimensional irreducible representation $V^\mu$, the set of weights $P^\mu$ that occur with positive multiplicity is saturated [15, §13.4], which means it is finite, invariant under the Weyl group $W_0$, and its hull is a convex polytope $\Pi$ in $E_0$ such that $P^\mu = \Pi \cap \tilde{Q}$, where $\tilde{Q} := Q + \mu$ is the coset of $Q$ in $P$ containing $\mu$.

We now look at the problem of estimating $|P^\mu|$ for a dominant integral weight $\mu$. This is the same as the number of points from the lattice $\tilde{Q}$ which are contained in $\Pi$. There is an extensive literature about the theory of lattice points in polyhedra and other convex bodies. In the past decade this seems to have been an area of intensive research and many new and interesting results were obtained mainly by means of a correspondence between convex polytopes and toric varieties [6], [36]. In this correspondence the number of lattice points in the polytope is expressed in terms of the Euler-Poincaré characteristic of the toric variety. Barvinok and Pommersheim [3] recently studied the problem of efficient enumeration of lattice points in polyhedra, and they introduced a polynomial-time algorithm for computing the number of those points for simple lattice polytopes. For the particular case of $P^\mu$, using results of Morelli [31], we could write down an explicit formula expressing $|P^\mu|$ as a polynomial of degree $n$ in the coordinates of $\mu$ with respect to the fundamental weights. Nevertheless, since we are only looking at the asymptotics, we can get an upper bound by inscribing $\Pi$ in a sphere of Euclidean radius $r = \|\mu\|$ and use an approximation for the number of lattice points contained therein. The literature on the circle and the sphere problems is also vast (we checked [17], [7] and [39]), and
curiously the former problem appears to be significantly harder than the latter. Here we will not need the full power of the recent developments in these areas.

It is well-known (cf. Landau, Hardy) that for a lattice \( L \) spanned by the linearly independent vectors \( v_1, \ldots, v_n \) in \( \mathbb{R}^n \) the counting function \( N_n(r) = \{ z \in L \mid \|z\| \leq r \} \) is asymptotic to \( \pi^{n/2} \Gamma(1 + \frac{1}{2} n)^{-1} r^n (\det V)^{-1} \), the volume of a sphere of radius \( r \) divided by the determinant of the matrix \( V \) whose columns are \( v_1, \ldots, v_n \). Therefore we have

\[
|P^\mu| \sim \frac{\pi^{n/2}}{\Gamma(1 + \frac{1}{2} n) \det(\alpha_1, \ldots, \alpha_n)} r^n.
\]

This suffices to show that \( |P^\mu| = O(r^n) \) for a fixed Lie algebra \( g \).

Since we want to determine the character table \( T_m \) we also need to estimate the number of dominant weights \( \mu \) of length less than or equal to that of \( \lambda \). The set \( P_m^+ \) of such weights is precisely the intersection of the cone of dominant weights \( P^+ \) with the ball \( \{ \mu \in P \mid \|\mu\| \leq m \} \). Using an argument similar to the one above we arrive at

\[
|P_m^+| \sim \frac{\pi^{n/2}}{|W_0| \Gamma(1 + \frac{1}{2} n) \det(\omega_1, \ldots, \omega_n)} m^n,
\]

where \( \omega_1, \ldots, \omega_n \) are the fundamental weights. This shows \( |P_m^+| = O(m^n) \). Finally we observe that \( |P^\lambda| \) is a uniform bound for \( |P^\mu| \) for all \( \mu \) of length less than or equal to that of \( \lambda \). Therefore \( |T_m| = O(m^{2n}) \), which proves both Proposition 1.1.1 and Corollary 1.1.2.

### 1.4 Study of Freudenthal’s algorithm

Let \( g, h, R_0, R_0^+, \Delta, P, P^\lambda \) and \( Q \) be as in section 1.2. If \( \alpha \) is a positive root then \( \alpha = \sum_{i=1}^n c_i \alpha_i \) for \( c_i \) in \( \mathbb{Z}_+ \) and we define the height of \( \alpha \) by \( \text{ht} \alpha = \sum_{i=1}^n c_i \). Now if \( \mu \) is in \( P^\lambda \) then \( \mu = \lambda - \sum_{i=1}^n d_i \alpha_i \) for \( d_i \) in \( \mathbb{Z}_+ \) and we define the level (or layer) of \( \mu \) (relative to \( \lambda \)) by \( \text{lv} \mu = \sum_{i=1}^n d_i \). We have \( \text{lv} \lambda = 0, \text{lv} (\mu - \alpha_1) = 1, \) etc, and if \( L_j = \{ \mu \in P^\lambda \mid \text{lv} \mu = j \} \), then \( P^\lambda = \bigsqcup_{j=0}^{2\text{lv}(-\lambda)} L_j \). Before we can apply Freudenthal’s formula to compute the multiplicity of a weight, we have to determine the weight system \( P^\lambda \). One way to accomplish this is by using the Dynkin’s layer method. We begin with the set \( L_0 = \{ \lambda \} \) and proceed inductively to construct \( L_{j+1} \) by considering all weights of the form \( \mu - \alpha_i \) for \( i = 1, \ldots, n \) and \( \mu \) in \( L_j \). To test whether \( \mu - \alpha_i \) lands in \( L_{j+1} \) we use
the fact that $P^\lambda$ is saturated and hence the $\alpha_i$-string $\mu - r\alpha_i, \mu - (r - 1)\alpha_i, \ldots, \mu + q\alpha_i$ through $\mu$, where $r - q = \langle \mu, \alpha_i \rangle$, is entirely contained in $P^\lambda$. Then it is enough to find the maximum $q$ such that $\mu + q\alpha_i$ is contained in one of the previously determined level sets $L_j, L_{j-1}, \ldots, L_0$ and compute $r = \langle \mu, \alpha_i \rangle + q$. If $r > 0$ then $\mu - \alpha_i$ belongs to $L_{j+1}$.

With the weight system $P^\lambda$ at hand we can compute each multiplicity $m_\lambda(\mu)$ using Freudenthal’s recursion formula [15]

$$
((\lambda + \rho, \lambda + \rho) - (\mu + \rho, \mu + \rho))m_\lambda(\mu) = 2 \sum_{\alpha \in R^+_0} \sum_{j=1}^\infty \langle \mu + j\alpha, \alpha \rangle m_\lambda(\mu + j\alpha). \quad (1.3)
$$

Using the fact that $m_\lambda(\lambda) = 1$, we can proceed by levels as in the Dynkin layer method, since for a given weight $\mu$ of level $j$, the right-hand side of this formula depends only on the multiplicities of weights of levels strictly smaller than $j$. Moreover, the constant on the left-hand side can be shown to be nonzero if $\lambda \neq \mu$ (cf. [15, §13.4, Lemma C]).

We remark that one can modify the above formula as in [30] so as to take advantage of the fact that certain weights $\mu$ have a nontrivial stabilizer and the multiplicities are constant along orbits of the Weyl group, but we are not pursuing this here because it would only affect the complexity by a constant factor depending on $|W_0|$.

---

**Input:** A dominant integral weight $\lambda$

**Output:** The character $\chi_\lambda$ in the monomial basis $M$

1. Use Dynkin’s Layer Method to determine $P^\lambda = L_0 \sqcup L_1 \sqcup \cdots$.
2. $m_\lambda(\lambda) \leftarrow 1$.
3. **for** $j = 0, 1, \ldots$ **do**
4. **for** all $\mu$ in $L_j$ **do**
5. Use formula (1.3) to compute the multiplicity $m_\lambda(\mu)$.
6. **end** for
7. **end** for
8. Form the character polynomial $\chi_\lambda = \sum_{\mu \in P^\lambda \cap P^+} m_\lambda(\mu)M_\mu$.

**Algorithm 1:** Freudenthal’s method

**Proof of 1.1.3.** It is easy to see that the Dynkin layer method can be run in $O(m^n)$
time. Moreover, the right-hand side of Freudenthal’s formula only involves weights lying along $\alpha$-strings $\mu + r\alpha$ for $r \in \mathbb{Z}_+$ and positive roots $\alpha$. These are evenly spaced points contained in lines through the sphere of radius $m = \|\lambda\|$ containing the weight system $P^\lambda$, hence the right-hand side involves at most $O(m)$ terms. Given our computing model, the complexity for this individual calculation is also $O(m)$. In order to compute the character, this work is to be repeated once for each coefficient, hence this can be done in $O(m^{n+1})$ time. Finally, for computing the character table $T_m$ this is to be repeated for each weight in $P_m^+$ which gives $O(m^{2n+1})$ time.

1.5 Sahi’s formula for weight multiplicities and characters

In this section we give a brief summary on the results of Sahi, Opdam and Heckman. The setup is similar to that in [35] and all results in this section can be found in [33] and [35], except for Proposition 1.5.8 which we prove here.

**Definition.** For each $\lambda \in P$, we define

1. 
\[ \tilde{\lambda} = \lambda + (1/2) \sum_{\alpha \in R_+^\ast} \varepsilon_{(\alpha^\vee, \lambda)} \alpha \]  
(1.4)

where for $t \in \mathbb{R}$, $\varepsilon_t$ is 1 if $t > 0$ and $-1$ otherwise.

2. Fix a reduced expression $s_i \cdots s_i$ for $w_{\lambda}$. Then for each $J = (j_1, \ldots, j_m)$ in $\{0, 1\}^m$, define $w_J := s_{i_1}^{1-j_1} \cdots s_{i_m}^{1-j_m}$. In other words, $w_J$ is $w_\lambda$ with some simple reflections $s_{j_i}$ deleted. For example, $w_{(1,1,\ldots,1)} = 1$ and $w_{(0,0,\ldots,0)} = w_\lambda$.

3. Define $c_J := c_1^{j_1} \cdots c_m^{j_m}$ where $c_j := (a_{i_j}, \lambda_{(j)})^{-1}$ and $\lambda_{(j)} = s_j \cdots s_1 \tilde{\lambda}$.

The formula for weight multiplicities of Sahi is given by the next theorem. One remarkable feature of this formula is to express the multiplicity $m_\lambda(\mu)$, an integer number, as a sum of certain positive rational numbers. Previously known formulas of this type, like Freudenthal’s, were sums involving only positive and negative integers.
Theorem 1.5.1 For $\lambda$ in $P^+$ and for any $\mu$ in $P^\lambda$, the multiplicity $m_{\lambda}(\mu)$ of $\mu$ in the irreducible representation $V^\lambda$ of $\mathfrak{g}$ is given by the formula

$$m_{\lambda}(\mu) = \frac{|W_0 \cdot \lambda|}{|W_0 \cdot \mu|} \sum_J c_J$$

(1.5)

where the sum ranges over all the multi-indices $J$ such that $w_J^{-1} \cdot \lambda$ is in $W_0 \cdot \mu$.

The positivity of the $c_J$ is a non-trivial fact and its proof corresponds to a substantial part of the work in [35, sec. 6]. This formula is equivalent to the following result. If we let $e^\mu$ denote the (real-valued) function $x \mapsto e^{(\mu, x)}$ on $E_0$, then $W$ acts on the $e^\mu$ via $s_i \cdot e^\mu = e^{s_i \cdot \mu}$. Recall that the character of $V^\lambda$ is the polynomial

$$\chi_\lambda := \sum_{\mu \in P} m_{\lambda}(\mu)e^\mu$$

(1.6)

Theorem 1.5.2 For $\lambda$ a dominant weight in $P$, we have the following identity for the character of $V^\lambda$:

$$\chi_\lambda = \frac{|W_0 \cdot \lambda|}{|W_0|} \sum_{w \in W_0} w(s_{i_m} + c_m) \cdots (s_{i_1} + c_1)e^\lambda$$

(1.7)

This theorem actually follows from a more general result, namely a similar formula for the generalized Jacobi polynomials $P_\lambda$ introduced by Heckman and Opdam. These polynomials are parametrized by root multiplicities $k$ which are functions on the roots with the property that $k_w \cdot \alpha = k_\alpha$, for all $\alpha$ in $R_0$ and all $w$ in $W_0$. For special root multiplicities, the $P_\lambda$ can be interpreted as the spherical function on a compact symmetric space.

It will follow immediately from the next theorem that $P_\lambda \to \chi_\lambda$ as $k \to 1$, provided we redefine $\tilde{\lambda}$ and the $c_J$ as follows.

Definition. For $k$ a root multiplicity, we redefine:

1'.

$$\tilde{\lambda} = \lambda + (1/2) \sum_{\alpha \in R_0^+} k_\alpha \varepsilon(\alpha, \lambda) \alpha;$$

(1.8)

3'. $c_J := k_i(a_{ij}, \tilde{\lambda}(j))^{-1}$, where $k_0 = k_\beta$ and $k_j = k_{\alpha_j}$ for $j \geq 1$. 

**Theorem 1.5.3** For \( \lambda \) a dominant weight in \( P \), \( \tilde{\lambda} \) and \( c_j \) as in (1’) and (3’), the generalized Jacobi polynomial \( P_\lambda \) is given by the same formula as in Theorem 1.5.2.

This is in fact the only currently known explicit formula for the polynomials \( P_\lambda \) and hence for the spherical function.

The polynomials \( P_\lambda \) are related to the eigenfunctions of the following differential-reflection operator. Let \( \mathbb{F} = \mathbb{R}(k_\alpha) \) be the field of rational functions in the parameters \( k_\alpha \) and let \( \mathcal{R} \) be the \( \mathbb{F} \)-span of \( \{e^\lambda \mid \lambda \in P\} \) as a \( W \)-module.

**Definition. 1.5.4** For \( y \in E_0 \), the Cherednik operator \( D_y \) is defined by

\[
D_y = \partial_y + \sum_{\alpha \in R_0^+} (y, \alpha) \frac{1}{1 - e^{-\alpha}} (1 - s_\alpha) - (y, \rho),
\]

(1.9)

where \( \rho := \frac{1}{2} \sum_{\alpha \in R_0^+} k_\alpha \alpha \).

Below we list some of the basic facts about the Cherednik operators, their eigenfunctions and their relation to the polynomials \( P_\lambda \).

**Proposition 1.5.5** We have the following.

1. The operators \( D_y \) act on \( \mathcal{R} \) and commute pairwise.

2. For \( i = 1, \ldots, n \), we have \( s_i D_y - D_{s_i y} s_i = -k_i (y, \alpha_i) \).

3. There is a basis \( \{E_\lambda \mid \lambda \in P\} \) of \( \mathcal{R} \) such that:

   (a) the coefficient of \( e^\lambda \) in \( E_\lambda \) is 1;

   (b) \( D_y E_\lambda = (y, \tilde{y}) E_\lambda \), where \( \tilde{y} \) is as in the introduction.

4. For \( \lambda \in P \), the Heckman-Opdam polynomial is given by

\[
P_\lambda = \frac{|W_0 \cdot \lambda|}{|W_0|} \sum_{w \in W_0} w E_\lambda.
\]

5. For \( i = 0, \ldots, n \), if \( s_i \cdot \lambda \neq \lambda \), then \( \tilde{s_i \cdot \lambda} = s_i \cdot \tilde{\lambda} \).

**Proposition 1.5.6** The polynomials \( E_\lambda \) satisfy the following recursions:
1. \( E_\lambda = e^\lambda \) for \( \lambda \in \mathcal{O} \);

2. if \( s_i \cdot \lambda \neq \lambda \), then \( (s_i + (k_i/(a_i, \overline{\lambda})))E_\lambda \) is a multiple of \( E_{s_i\lambda} \).

**Corollary 1.5.7** For \( \lambda \in P \), and \( c_i \) as defined above, we have

\[
E_\lambda = (s_{i_m} + c_m) \cdots (s_{i_1} + c_1) e^\lambda
\]

(1.10)

At the core of the new algorithm for computing the character is the following recursion formula. (see Procedure 3).

**Proposition 1.5.8** For \( k \) a root multiplicity function, if \( \lambda \in P \) is such that \( (\lambda, a_j) < 0 \) for some \( j = 0, 1, \ldots, n \) then the following recursion holds:

\[
E_\lambda = \left( s_j + \frac{k_j}{(a_j, s_j \cdot \lambda)} \right) E_{s_j \cdot \lambda}.
\]

(1.11)

Otherwise, if \( (\lambda, a_j) \geq 0 \), then \( E_\lambda = e^\lambda \).

**Proof.** If \( (\lambda, a_j) \geq 0 \) for all \( j = 0, 1, \ldots, n \) then \( \lambda \in \mathcal{O} \) and \( E_\lambda = e^\lambda \). Now, fix a reduced expression \( s_{i_1} \cdots s_{i_m} \) for \( w_\lambda \). If \( (\lambda, a_j) < 0 \) we can choose \( i_m = j \) (cf. [16, §4.6]). In this case, we observe that \( s_j \cdot \lambda \) has length one less than that of \( \lambda \), so repeated applications of the formula will eventually come to a stop with a weight in \( \mathcal{O} \). In particular, \( s_j \cdot \lambda \neq \lambda \), and Proposition 1.5.6(2) gives us \( E_\lambda = (s_j + c_m)E_{s_j \cdot \lambda} \) (the coefficient of \( e^\lambda \) being 1 in both expressions). But \( c_m = k_j(a_j, s_j \cdot \overline{\lambda})^{-1} \) and \( \lambda_{(m)} = s_{i_{m-1}} \cdots s_{i_1} \cdot \overline{\lambda} = s_j \cdot \lambda \). Also by Proposition 1.5.5(5) we have \( s_j \cdot \overline{\lambda} = s_j \cdot \overline{\lambda} \), so we get \( c_m = k_j(a_j, s_j \cdot \overline{\lambda})^{-1} \), as required.

Finally we remark that when \( k_\alpha \to 1 \) for all roots \( \alpha \), Theorem 1.5.2 and Corollary 1.5.7 together imply that the symmetrization of \( E_\lambda \) under the action of \( W_0 \) times a suitable constant is precisely \( \chi_\lambda \), i.e.,

\[
\chi_\lambda = c \sum_{w \in W_0} w \cdot E_\lambda.
\]

(1.12)

From the observation in Proposition 1.5.5(3a) that the coefficient of \( e^\lambda \) in \( E_\lambda \) is 1, we deduce that the constant \( c \) is exactly \( |W_0 \cdot \lambda|/|W_0| \).
1.6 Study of the new algorithm

In this section we present the new algorithm and proceed with its complexity analysis. For this purpose we break the main algorithm down into smaller procedures which can be easily analyzed. Its core is Procedure 3 which implements the recursion formula in Proposition 1.5.8.

The factor $|W_0|/|W_0 \cdot \mu|$ is the size of the stabilizer of $\mu$ in $W_0$. This number can be determined from the Dynkin diagram of $\mathfrak{g}$, using the fact that the stabilizer of a weight is generated by the simple reflections it contains [16, §1.12].

**Input:** A dominant integral weight $\lambda$ in $P^+$.

**Output:** The character $\chi_\lambda$ in the monomial basis $M$.

1. Call Procedure 3 to compute $E_\lambda$.
2. for all $e^\mu$ in $E_\lambda$ do
3. Let $\overline{\mu}$ be the dominant weight conjugate to $\mu$ under $W_0$.
4. Determine $|\text{Stab}_{W_0}\mu|$ by looking at the simple reflections fixing $\mu$ and at the corresponding subdiagram of the Dynkin diagram.
5. Replace $e^\mu$ with $|\text{Stab}_{W_0}\mu| \cdot M_\overline{\mu}$, where $M_\overline{\mu}$ stands for the monomial symmetric polynomial of $\overline{\mu}$.
6. end for
7. Divide the resulting polynomial by $|\text{Stab}_{W_0}\lambda|$.
8. Return

$$\chi_\lambda = \sum_{\mu \in P_+^\lambda} m_\lambda(\mu)M_\mu.$$ 

**Algorithm 2:** Compute the character $\chi_\lambda$

The following result gives running times for computing a single $E_\lambda$ with and without the assumption that previously computed $E_\mu$’s are known.

**Theorem 1.6.1** Given a (not necessarily dominant) weight $\lambda$ of length $m = l(w_\lambda)$ in the weight lattice $P$ of rank $n$, Procedure 3 computes a single polynomial $E_\lambda$ in $O(m^{n+1})$ time. However, assuming that the polynomials $E_\mu$ for weights $\mu$ of length less than that
Input: Any weight $\lambda$ in the weight lattice $P$.

Output: The polynomial $E_\lambda$ as a sum of monomials $e^\mu$.

1: if $(\lambda, a_j) < 0$ for some $j = 0, 1, \ldots, n$ then
2: Recursively call Procedure 3 to compute $E_{s_j \cdot \lambda}$.
3: Apply Operator $(s_j + c)$ to $E_{s_j \cdot \lambda}$ where $c = \frac{k_j}{(a_j, s_j \cdot \lambda)}$
4: Return $E_\lambda$.
5: else
6: Return $e^\lambda$.
7: end if

Procedure 3: Compute the polynomial $E_\lambda$

of $\lambda$ have been previously determined, Procedure 3 computes $E_\lambda$ in $O(m^n)$ time.

Proof. The correctness of the algorithm is clear from Proposition 1.5.8 and the subsequent remarks. Observe that the number of terms in $E_\lambda$ is at most $O(m^n)$ since each monomial in this polynomial comes from a weight in $P^\lambda$.

Let $T(m)$ be the cost for computing $E_\lambda$. If $\lambda$ is not in $O$, then $(a_j, \lambda) < 0$ for some $0 \leq j \leq n$ and $s_j \cdot \lambda$ has length $m - 1$, which enables us to use recursion (1.11) to compute $E_\lambda$ by inductively using Procedure 3 with $\lambda$ replaced with $s_j \cdot \lambda$ and then applying the reflection operator $(s_j + c)$ to $E_{s_j \cdot \lambda}$. The total cost of this operation is $T(m) = T(m - 1) + O(m^n)$, therefore $T(m) = O(m^{n+1})$. This proves the first assertion.

On the other hand, if $E_{s_j \cdot \lambda}$ is known, its computation cost drops down to $O(1)$ and hence $T(m) = O(m^n)$. This finishes the proof.

We now apply this result to prove one of the main theorems stated in the introduction.

Proof of 1.1.4. In Algorithm 2, step 1 can be done in either $O(m^n)$ or $O(m^{n+1})$ time depending on whether or not a cache mechanism is used, i.e. whether of not previously computed polynomials $E_\mu$ are kept in memory as needed for future calculations. Steps 2-6 are to be repeated once for each of the $O(m^n)$ terms in $E_\lambda$ while Steps 3 and 4 can be done in $O(|W_0|)$ time or better and Step 5 can be done in $O(1)$ time, hence Steps 2-6 can be completed in $O(m^n)$ time. Step 7 can also be done in $O(m^n)$ time. Observe that
polynomial simplifications are likely to occur in this step because different weights $\mu$ and $\nu$ might be conjugate to a same dominant weight. Therefore the total running time of the algorithm is accordingly either $O(m^n)$ or $O(m^{n+1})$ time.

Of course, in computing the full character table $T_m$ we can take advantage of the previously determined $E_\mu$’s, hence each character $\chi_\mu$ for $\mu$ of length $k$ can be computed in $O(k^n)$ time. Summing this up for all of its $O(m^n)$ characters results in a time complexity of $O(m^{2n})$, which completes the proof.

\section{1.7 Computational considerations}

The algorithms described in this chapter were first implemented and tested in MAPLE. For a better performance and convenience of analysis, they were subsequently implemented as stand-alone applications written in C++. The correctness of these programs was verified by comparing their output with published tables of multiplicities \cite{4} and also by using the \texttt{LiE} package \cite{40}. Their source code will soon be available for download from the author’s website.

Since the coefficients of $E_\lambda$ are rational numbers, it would seem at first that we should use rational arithmetic. However, we can avoid this by choosing a suitable normalization, namely $F_\lambda = c_\lambda E_\lambda$, where $c_\lambda$ is defined as above. Sahi \cite{35} has shown the $F_\lambda$’s have only integer coefficients. This comes at expense of working with possibly large integers. We used this approach in our C++ implementation, with the aid of a public-domain C++ package that allows for multi-precision integer arithmetic.

Figure 1.1 shows a comparative performance graph of the methods we implemented for computing the character table $T_m$ of the Lie algebra $\mathfrak{sl}_3$. The horizontal axis is the radius $m$ while the vertical axis shows the cumulative number of arithmetic operations. As expected, the advantage of Sahi’s method over Freudenthal’s is evident from that graph.
Figure 1.1: Comparative performance for type $A_2$
Chapter 2

Recursion formulas for characters and multiplicities of irreducible representations of simple Lie algebras

In this chapter we give new recursion formulas for characters and weight multiplicities. These extend to all classical Lie algebras the well-known recursion

\[ \chi_\lambda = \Theta_\lambda + \chi_{\lambda - \rho}, \]  

(2.1)

for the Lie algebra \( A_2 \), where \( \Theta_\lambda \) is the girdle, \( \chi_\mu \) is the signed character, and \( \rho \) is half the sum of all positive roots (precise definitions are given below).

We also give closed formulas for the number of weights that occur in any finite dimensional irreducible representation. For instance, if \( \lambda = (a, b) \) is the highest weight of an irreducible representation of \( A_2 \) expressed in the weight basis and \( P_\lambda \) is the corresponding weight system, then we will show that the total number of weights that occur is given by the following quadratic polynomial in \( a \) and \( b \) with rational coefficients:

\[ |P_\lambda| = \left( \frac{a + b + 2}{2} \right) + ab. \]  

(2.2)

2.1 Introduction

Let \( \mathfrak{g} \) be a complex simple Lie algebra of rank \( n \) over \( \mathbb{C} \), \( \mathfrak{h} \) a Cartan subalgebra of \( \mathfrak{g} \), \( \mathfrak{h}^* \) its dual space, \( R \) the root system of \( \mathfrak{g} \) in \( \mathfrak{h}^* \), \( \Delta = \{ \alpha_1, \ldots, \alpha_n \} \) a choice of simple roots in \( R \), \( R^+ \) the positive roots, and \( R^- = -R^+ \) the negative roots. The \( \mathbb{R} \)-span of \( \Delta \) in \( \mathfrak{h}^* \) is an Euclidean space \( E \) where the symmetric non-degenerate bilinear form \( \langle , \rangle \) on \( E \) can be chosen up to a scalar factor to be the Killing form of \( \mathfrak{h} \). As usual, we define the root lattice \( Q \) of \( \mathfrak{g} \) as the \( \mathbb{Z} \)-span of \( \Delta \) and define the the co-roots \( \alpha_i^\vee \) as \( 2\alpha_i/\langle \alpha_i, \alpha_i \rangle \) for each \( \alpha_i \) in \( R \). The vectors \( \omega_1, \ldots, \omega_n \) of \( E \) defined by \( \langle \omega_i, \alpha_i^\vee \rangle = \delta_{ij} \), where \( \delta_{ij} \) is the Kronecker
delta, are called fundamental weights of \( g \) and their \( \mathbb{Z} \)-span is a lattice \( P \) in \( E \) called the weight lattice of \( g \). This contains \( Q \) as a sublattice of finite index, known as the index of connection, which is equal to the determinant \( d \) of the Cartan matrix \( C = [\langle \alpha_i, \alpha_j^\vee \rangle] \). The set \( P^+ \) containing all weights \( \mu \) in \( P \) satisfying \( \langle \mu, \alpha^\vee \rangle \geq 0 \) for all \( \alpha \) in \( \Delta \) forms a “convex cone” in \( P \) called the cone of dominant weights. The Weyl group \( W \) of \( R \) is defined as the subgroup of \( GL(E) \) generated by the simple reflections \( s_i(\lambda) = \lambda - \langle \lambda, \alpha^\vee_i \rangle \alpha_i \) for \( i = 1, \ldots, n \). It is a Coxeter group with generators \( s_i \), so we have the notion of length \( l(w) \) for all \( w \) in \( W \), namely the length of a shortest expression of \( w \) in terms of the generators. All these definitions can be found for example in [15] or [16].

For each dominant (integral) weight \( \lambda \), the irreducible representation \( V^\lambda \) of highest weight \( \lambda \) is finite dimensional and its weights form a saturated subset \( P^\lambda \) of \( P \) in the sense of [15, §13.4]. In particular, this means \( P^\lambda \) is finite and invariant under \( W \). Let \( \mathbb{C}[P] \) be the group algebra with basis \( e^\mu \) for \( \mu \) in \( P \). We define the (formal) character of \( V^\lambda \) as the sum

\[
\chi_{\lambda} = \sum_{\mu \in P^\lambda} m_{\lambda}(\mu) e^\mu,
\]

where the coefficient \( m_{\lambda}(\mu) \) is the multiplicity of the weight \( \mu \) in the representation \( V^\lambda \), i.e. the dimension of the weight space

\[
V^\lambda(\mu) = \{ v \in V^\lambda | h \cdot v = \mu(h)v, \text{for all } h \text{ in } h \}.
\]

Our recursion formulas will be expressed as a signed integral sum of characters and a girdle. The girdle is an exponential sum similar to a character, except by having all nonzero coefficients equal to 1. For convenience, we will also work with signed versions of characters that are defined for arbitrary weights. These notions are formalized in the following way.

**Definition.** 1. For each weight \( \lambda \), let \( w_{\lambda} \) be the unique shortest Weyl group element such that \( w_{\lambda}(\lambda + \rho) \) is dominant. We set \( \overline{\lambda} = w_{\lambda}(\lambda + \rho) - \rho \). Note that if \( \lambda \) is dominant then \( w_{\lambda} = 1 \) and \( \overline{\lambda} = \lambda \), and if \( \lambda + \rho \) is regular, i.e., \( \langle \lambda + \rho, \alpha \rangle \neq 0 \) for \( \alpha \in R^+ \), then \( \overline{\lambda} \) is dominant.
2. For any weight $\lambda$ let

$$
\varepsilon_\lambda = \begin{cases} 
(-1)^{l(w_\lambda)}, & \text{if } \lambda + \rho \text{ is regular} \\
0, & \text{otherwise}
\end{cases}
$$

3. We redefine the character $\chi_\lambda$ to be a signed character. When $\lambda$ is dominant, we retain the previous definition for $\chi_\lambda$ which we shall call actual characters. Otherwise if $\lambda$ is not dominant, then we define $\chi_\lambda = \varepsilon_\lambda \chi_{\cal T}$. 

4. Accordingly, we extend the notion of multiplicity. If $\lambda$ is dominant, we retain the above definition for $m_\lambda(\mu)$. Otherwise, if $\lambda$ is not dominant, then we let $m_\lambda(\mu) = \varepsilon_\lambda m_{\cal T}(\mu)$. In any case we set $m_\lambda(\mu) = 0$ if $\mu$ is not in $P^\lambda$. 

5. For each dominant weight $\lambda$ we define the girdle as

$$
\Theta_\lambda = \sum_{\mu \in P^\lambda} e^\mu.
$$

Since $|P^\lambda|$ is invariant under $W$, so is the girdle, hence it is natural to ask for its expansion in terms of characters. Finding this explicitly is one of our main results, Theorem 2.1.1 below.

For simplicity of notation, we will denote by $\Delta' = R^+ \setminus \Delta$ the set of all positive non-simple roots, and for any subset $\Phi$ of $E$ we will indicate by $\langle \Phi \rangle$ the sum of all the elements in $\Phi$.

For each subset $\Phi$ of $\Delta'$, let $\mathcal{F}_\Phi$ be the collection of all subsets $\Psi$ of $\Delta'$ such that $\langle \Psi \rangle = \langle \Phi \rangle$. We can partition $\mathcal{F}_\Phi$ into $\mathcal{F}^0_\Phi$ and $\mathcal{F}^1_\Phi$ according to whether the subsets in each one contain an even or odd number of elements respectively. The number $c_{\langle \Phi \rangle} = |\mathcal{F}^0_\Phi| - |\mathcal{F}^1_\Phi|$ is the difference between the number of all possible ways of writing $\langle \Phi \rangle$ as an even sum of roots in $\Delta'$ and the number of all possible ways of writing $\langle \Phi \rangle$ as an odd sum of roots in $\Delta'$ without repetitions. Let $\mathcal{F}$ be the set of all $\langle \Phi \rangle$ for all subsets $\Phi$ of $\Delta'$.

The expression $\prod_{\alpha \in \Delta'} (1 - e^{-\alpha})$ is going to play an important role in deriving our recursions. By multiplying out its factors and using the above notation for $\langle \Phi \rangle$ and $c_{\langle \Phi \rangle}$, we can rewrite it as

$$
\prod_{\alpha \in \Delta'} (1 - e^{-\alpha}) = 1 + \sum_{\Phi} (-1)^{|\Phi|} e^{-\langle \Phi \rangle},
$$

(2.3)
where the summation is over all nonempty subsets $\Phi$ of $\Delta'$. By grouping the terms in this summation, we can further write

$$\prod_{\alpha \in \Delta'} (1 - e^{-\alpha}) = 1 + \sum_{\langle \Phi \rangle \in \mathcal{F}} c_{\langle \Phi \rangle} e^{-\langle \Phi \rangle}. \tag{2.4}$$

We are now ready to state our main results.

**Theorem 2.1.1 (Recursion Formula for Characters)** If $\lambda$ is a dominant weight and $c_{\Phi}$ is defined as above then we have the following expansion for the girdle:

$$\Theta_\lambda = \chi_\lambda + \sum_{\langle \Phi \rangle \in \mathcal{F}} c_{\langle \Phi \rangle} \chi_{\lambda - \langle \Phi \rangle}, \tag{2.5}$$

Furthermore, this is a linear recursion expressing the actual character $\chi_\lambda$ as a sum of the girdle with “lower” signed characters.

We shall prove identity (2.5) by showing that both sides are equal to

$$\sum_{w \in W} w \cdot \frac{e^\lambda}{\prod_{\alpha \in \Delta} (1 - e^{-\alpha})}. \tag{2.6}$$

See Theorems 2.1.5 and 2.1.6 below. Also, it is not obvious why it works as a recursion. This will be proved in section 2.3. Immediate corollaries are a recursion formula for multiplicities and two different exact formulas for $|P^\lambda|$.

**Corollary 2.1.2 (Recursion Formula for Multiplicities)** If $\lambda$ is dominant and $\mu$ is any weight in $P^\lambda$, then

$$m_\lambda(\mu) = 1 - \sum_{\langle \Phi \rangle \in \mathcal{F}} c_{\langle \Phi \rangle} m_{\lambda - \langle \Phi \rangle}(\mu), \tag{2.7}$$

and this is a linear recursion expressing the multiplicity $m_\lambda(\mu)$ as an integral linear combination of multiplicities of the same weight $\mu$ for “lower” representations.

This follows by simply comparing coefficients on both sides of (2.5).

**Corollary 2.1.3 (First exact formula for $|P^\lambda|$)** The number of weights in $P^\lambda$ is equal to

$$\dim V^\lambda + \sum_{\langle \Phi \rangle \in \mathcal{F}} \varepsilon_{\lambda - \langle \Phi \rangle} c_{\langle \Phi \rangle} \dim V^{\lambda - \langle \Phi \rangle}. \tag{2.8}$$
We can show this by looking at \( e^\mu \) as a function on \( E \) defined by \( e^\mu(x) := e^{\langle \mu, x \rangle} \). Then we just evaluate both sides of (2.5) at 0. The left-hand side then evaluates to \( |P^\lambda| \) while the right-hand side evaluates to (2.8).

For the next corollary we need the notion of *Todd polynomials* \( T_k \) [14]. These are homogeneous polynomials of degree \( k \) defined by the expansion

\[
\prod_{i=1}^n \frac{tx_i}{1 - e^{-tx_i}} = \sum_{k=0}^{\infty} T_k(x_1, x_2, \ldots, x_n) t^k
\]  

(2.9)

**Corollary 2.1.4 (Second exact formula for \( |P^\lambda| \))** The number of weights in \( P^\lambda \) is equal to

\[
\sum_{\mathbf{w} \in W} \frac{1}{\Pi_{i=1}^n \langle \rho, w\alpha_i \rangle} \sum_{j=0}^{n} \frac{\langle \rho, w\lambda \rangle^j}{j!} T_{n-j}(\langle \rho, w\alpha_1 \rangle, \ldots, \langle \rho, w\alpha_n \rangle)
\]  

(2.10)

Furthermore, if \( \lambda = (m_1, \ldots, m_n) \) in the weight basis then \( |P^\lambda| \) is a polynomial of degree \( n \) in the \( m_1, \ldots, m_n \).

This will be shown in section 2.2. It will actually follow from the aforementioned identity between the girdle \( \Theta^\lambda \) and (2.6).

In what follows, we shall be regarding certain functions as rational functions on the (formal) variables \( e^{-\alpha_i} \), i.e, as elements in the field \( \mathbb{C}(e^{-\alpha_1}, \ldots, e^{-\alpha_n}) \). In order to simplify the notation, we shall rename \( e^{-\alpha_i} \) to \( x_i \). This can be formalized by means of an appropriate homomorphism. Moreover, we shall be looking at particular expansions of these rational functions as Laurent series in the \( x_i \). For this purpose we establish the following expansion convention:

\[
(1 - e^{-\alpha})^{-1} = 1 + e^{-\alpha} + e^{-2\alpha} + \cdots, \text{ for } \alpha > 0,
\]  

(2.11)

and

\[
(1 - e^{-\alpha})^{-1} = -e^\alpha - e^{2\alpha} - \cdots, \text{ for } \alpha < 0.
\]

(2.12)

**Theorem 2.1.5** Let \( \lambda \) be a dominant weight. Then with the above expansion convention, the following identity between rational functions holds:

\[
\sum_{\mathbf{w} \in W} \mathbf{w} \cdot \frac{e^\lambda}{\Pi_{\alpha \in \Delta}(1 - e^{-\alpha})} = \Theta^\lambda.
\]  

(2.13)
This will follow from Propositions 2.2.2 and 2.2.3.

**Theorem 2.1.6** If \( \lambda \) is a dominant weight, then the following identity between rational functions holds:

\[
\sum_{w \in W} w \cdot \frac{e^{\lambda}}{\prod_{\alpha \in \Delta} (1 - e^{-\alpha})} = \chi_{\lambda} + \sum_{\langle \Phi \rangle \in F} c_{\langle \Phi \rangle} \chi_{\lambda - \langle \Phi \rangle}.
\]  

(2.14)

This will be a consequence of the Weyl Character Formula for signed characters to be introduced below.

**Example 2.1.7** For type \( A_2 \), the only positive nonsimple root is \( \alpha_1 + \alpha_2 = \rho \), thus identity (2.5) becomes the well-known recursion (2.1).

**Example 2.1.8** For type \( B_2 \), the positive nonsimple roots are \( \alpha_1 + \alpha_2 \) and \( \alpha_1 + 2\alpha_2 \). Thus identity (2.5) becomes the recursion

\[
\chi_{\lambda} = \Theta_{\lambda} + \chi_{\lambda - \alpha_1 - \alpha_2} + \chi_{\lambda - \alpha_1 - 2\alpha_2} - \chi_{\lambda - 2\alpha_1 - 3\alpha_2}.
\]  

(2.15)

We will now use it to find the actual character \( \chi_{\lambda} \) for \( \lambda = \rho \). As usual, we work with the fundamental weight basis, so that each weight has integer coordinates. In this basis, \( \rho = (1,1) \), and the four positive roots are

\[
\alpha_1 = (-2,2), \quad \alpha_2 = (2,-1), \quad \alpha_3 = (1,0), \quad \text{and} \quad \alpha_4 = (0,2),
\]

where the first two are simple, so that \( \langle \Phi \rangle \) ranges over the set \( \{(1,0), (0,2), (1,2)\} \) and \( c_{\langle \Phi \rangle} \) is respectively one of \( \{-1, -1, +1\} \). It is convenient to express the character as a sum of orbit-sums \( M_{\mu} = \frac{|W_{\mu}|}{|W|} \sum_{w \in W} e^{w_{\mu}} \). Since \( P^{(1,1)} \cap P^+ = \{(1,1), (0,1)\} \), we have

\[
\Theta_{(1,1)} = M_{(1,1)} + M_{(0,1)}.
\]

Now \( (1,1) - \langle \Phi \rangle + \rho \) is regular only for \( \langle \Phi \rangle = (0,1) \), so formula (2.15) becomes simply

\[
\chi_{(1,1)} = \Theta_{(1,1)} + \chi_{(0,1)}.
\]

Similar considerations (or just the fact that \( (0,1) \) is minuscule) lead to \( \chi_{(0,1)} = M_{(0,1)} \), therefore

\[
\chi_{(1,1)} = M_{(1,1)} + 2M_{(0,1)}.
\]
By using formula (2.15) we can quickly build-up a character table for $B_2$ (see table 2.1). Since the determinant of the Cartan matrix for $B_2$ is 2, we have two conjugacy classes of weights there labeled *Class 0* and *Class 1*.

**Example 2.1.9** Using the recursion formula (2.7) for multiplicities and the data on table 2.1, we can for instance compute $m_{(5,0)}(0,0)$ directly since

$$m_{(5,0)} = 1 + m_{(4,0)} + m_{(5,-2)} + m_{(4,-2)}.$$  

It turns out that $w_{(5,-2)}((5, -2) + \rho) = (4,0) + \rho$ and $w_{(4,-2)}(4, -2) + \rho = (3,0) + \rho$, with $(-1)^{w_{(5,-2)}} = (-1)^{w_{(4,-2)}} = -1$, hence by definition $m_{(5,-2)} = -m_{(4,0)}$ and $m_{(4,-2)} = -m_{(3,0)}$, and we get

$$m_{(5,0)}(0,0) = 1 + 3 + (-3) - (-2) = 3.$$  

**Example 2.1.10** It is well-known that the zero-weight space of a representation of $\mathfrak{g}$ is also a representation for the Weyl group $W$ of $\mathfrak{g}$. The dimension of such a representation is equal to the multiplicity of the zero-weight space. In this example, we will obtain a closed formula for the dimension of a zero-weight space representation of $W$ for the Lie algebra $B_2$. These are indexed by the highest weights $\lambda$ and hence can be indexed by any two nonnegative integers $i$ and $j$ such that $\lambda = (i, 2j)$ in the weight basis.

**Proposition 2.1.11** For any two nonnegative integers $i$ and $j$, the dimension of the zero-weight space representation $(i, 2j)$ of $W$ is equal to

$$m_{(i,2j)}(0,0) = \frac{1 + (-1)^i}{4} + \frac{(i + 1)(2j + 1)}{2}. \quad (2.16)$$

For simplicity we set $a_{i,2j} = m_{(i,2j)}(0,0)$. We begin with the special cases of $a_{i,0}$ and $a_{0,2j}$.

For $\lambda = (i, 0)$, observe that $\lambda - \beta$ is one of $(i - 1, 0)$, $(i, -2)$ and $(i - 1, -2)$. Assuming that $i > 1$, the former is dominant but not the latter two. However, we can readily see that $(i, -2) + \rho$ is conjugate to $(i - 1, 0) + \rho$, and $(i - 1, -2) + \rho$ is conjugate to $(i - 2, 0)$ via the simple reflection $s_2$, so we get

$$a_{i,0} = 1 + a_{i-2,0}, \text{ for } i > 1,$$
Table 2.1: Character table for the Lie algebra $B_2$

<table>
<thead>
<tr>
<th>Class 0</th>
<th>00</th>
<th>10</th>
<th>02</th>
<th>20</th>
<th>12</th>
<th>30</th>
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and \( a_{0,0} = a_{10} = 1 \). This easy recurrence has generating function \( E(x) = \frac{1}{(1 + x)(1 - x)^2} \).

Similarly, the recursion for \( a_{0,j} \) gives the generating function \( F(y) = \frac{1}{(1 - y^2)^2} \). We can now construct the generating function \( G(x, y) \) for \( a_{i,2j} \). First we write the recurrence for \( a_{i+1,2j+2} \) and rearrange the terms obtaining

\[
a_{i,2j} + a_{i+1,2j+2} - a_{i,2j+2} - a_{i+1,2j} = 1.
\]

Then we multiply all terms by \( x^i y^{2j} \) and sum over all \( i \geq 0 \) and all \( j \geq 0 \) to obtaining

\[
\sum_{i,j \geq 0} a_{i,2j} x^i y^{2j} + \frac{1}{xy^2} \sum_{i,j \geq 0} a_{i+1,2j+2} x^{i+1} y^{2j+2} - \frac{1}{y^2} \sum_{i,j \geq 0} a_{i,2j+2} x^i y^{2j+2} - \frac{1}{x} \sum_{i,j \geq 0} a_{i+1,2j} x^{i+1} y^{2j} = \sum_{i,j \geq 0} x^i y^{2j}.
\]

Substituting for \( E(x) \) and \( F(y) \) we get

\[
G(x, y) + \frac{1}{xy^2}(G(x, y) - E(x) - F(y) + 1) - \frac{1}{y^2}(G(x, y) - E(x)) - \frac{1}{x}(G(x, y) - F(y)) = \frac{1}{(1 - x)(1 - y^2)^2},
\]

which resolves to

\[
G(x, y) = \frac{1 + xy^2}{(1 - x)(1 - x^2)(1 - y^2)^2}.
\]

Using an easy method of Ehrhart [10, Chap. II, §5] applied to this generating function, we can solve the recursion exactly for \( a_{i,2j} \) as follows. We first decompose \( G(x, y) \) as

\[
\frac{1}{4(1 + x)(1 - y^2)} + \frac{1}{4(1 - x)(1 - y^2)} + \frac{1 + y^2}{2(1 - x)^2(1 - y^2)^2}.
\]

(2.17)

Then we identify its factors as geometric series

\[
\frac{1}{4} \sum_{i,j=0}^\infty ((-1)^i + 1)x^i y^{2j} + \frac{1}{2} \sum_{i,j=0}^\infty ((i + 1)(j + 1) + (i + 1)j)x^i y^{2j}.
\]

(2.18)

and combine the coefficients to obtain the aforementioned expression for \( a_{i,2j} \).

We remark that the generating function \( G(x, y) \) is not new. Using different methods, Doković [32] gave explicit generating functions for all classical Lie algebras of rank 2, although apparently he did not attempt at solving them explicitly.
2.2 Laurent expansions

This section deals with proving our main results. It will require a few more definitions and lemmas. For each \( w \) in \( W \) let \( \Phi_w = -(wR^+ \cap R^-) \) be the set of all positive roots that get sent to negative roots by \( w \) and let \( \Delta_w = \Phi_w \cap \Delta \). We recall that if \( w \neq 1 \) then \( \Phi_w \) contains simple roots and hence \( \Delta_w \) is nonempty. We also define the complement \( \Delta'_w = \Delta \setminus \Delta_w \).

Let \( K_{w\lambda} \) denote the half-open “cone” generated by the linearly independent vectors \( w\alpha (\alpha \in \Delta_w) \) and \( -w\alpha (\alpha \in \Delta'_w) \) with vertex at \( w\lambda \), namely the set

\[
K_{w\lambda} = \{ w\lambda + \sum_{\alpha \in \Delta_w} k_{\alpha} w\alpha - \sum_{\alpha \in \Delta'_w} k_{\alpha} w\alpha \},
\]

where the \( k_{\alpha} \) are strictly positive integers for \( \alpha \in \Delta_w \), and non-negative integers for \( \alpha \in \Delta'_w \).

**Lemma 2.2.1** Let \( \lambda \) be a dominant weight. If \( \mu \) is a weight in \( P^\lambda \) and \( \mu \in K_{w\lambda} \) then \( w = 1 \).

**Proof.** In effect, it is well known that \( \mu \in P^\lambda \) if and only if \( w\mu \leq \lambda \) for all \( w \) in \( W \), hence if and only if \( \lambda - w\mu = \sum_{\alpha \in \Delta} j_\alpha(w)\alpha \) where the \( j_\alpha(w) \) are non-negative integers. Therefore \( \mu \in P^\lambda \) if and only if

\[
\mu = w\lambda + \sum_{\alpha \in \Delta_w} (-j_\alpha(w^{-1})) w\alpha - \sum_{\alpha \in \Delta'_w} j_\alpha(w^{-1}) w\alpha,
\]

which clearly cannot belong to \( K_{w\lambda} \) unless \( w = 1 \). \( \square \)

Below, we shall use the notation \( \delta_{w\lambda} \) for the characteristic function of \( K_{w\lambda} \), so that \( \delta_{w\lambda}(\mu) = 1 \) when \( \mu \in K_{w\lambda} \) and it is zero otherwise. As a first step in proving Theorem 2.1.5 we have the following proposition.

**Proposition 2.2.2** Let the series \( \sum_{\mu \in P} c_{\lambda\mu} e^\mu \) be the formal expansion of (2.6) according to the convention established by (2.11) and (2.12). Then the coefficient of \( e^\mu \) in that expansion is equal to

\[
c_{\lambda\mu} = \sum_{w \in W} (-1)^{|\Delta_w|} \delta_{w\lambda}(\mu).
\]

Furthermore, if \( \mu \in P^\lambda \) then \( c_{\lambda\mu} = 1 \).
Proof. Since $\Delta$ is linearly independent, given $w$ and $\mu$, we can solve the equation

$$w\lambda + \sum_{\alpha \in \Delta_w} (k_\alpha + 1)w\alpha - \sum_{\alpha \in \Delta'_w} k_\alpha w\alpha = \mu$$

uniquely for $k = (k_\alpha)$. Now a straightforward formal calculation shows that:

$$\sum_{w \in W} w \cdot e^{\lambda} \prod_{\alpha \in \Delta} (1 - e^{-\alpha}^{-1}) = \sum_{w \in W} e^{w\lambda} \prod_{\alpha \in \Delta_w} \left(1 - e^{-w\alpha}^{-1}\right) \prod_{\alpha \in \Delta'_w} \left(1 - e^{-w\alpha}^{-1}\right)$$

$$= \sum_{w \in W} (-1)^{|\Delta_w|} \sum_{k_\alpha \in \mathbb{Z}^{|\Delta_w|}} \exp(w\lambda + \sum_{\alpha \in \Delta_w} (k_\alpha + 1)w\alpha - \sum_{\alpha \in \Delta'_w} k_\alpha w\alpha)$$

$$= \sum_{w \in W} (-1)^{|\Delta_w|} \sum_{\mu \in K_{w\lambda}} e^{\mu}$$

$$= \sum_{\mu \in P} \left(\sum_{w \in W} (-1)^{|\Delta_w|} \delta_{w\lambda}(\mu)\right) e^{\mu}.$$  

The second statement follows from the previous lemma.

After this lemma we know that formally

$$\sum_{w \in W} w \cdot e^{\lambda} \prod_{\alpha \in \Delta} (1 - e^{-\alpha}^{-1}) = \Theta_{w\lambda} + \sum_{\mu \in P \setminus P_{w\lambda}} c_{\lambda \mu} e^{\mu},$$  \hspace{1cm} (2.19)

hence to complete the proof of Theorem 2.1.5 all we need to do is to check the following proposition.

**Proposition 2.2.3** If $\mu$ is not in $P_{w\lambda}$, then

$$c_{\lambda \mu} = \sum_{w \in W} (-1)^{|\Delta_w|} \delta_{w\lambda}(\mu) = 0.$$  

If we can show that the series on the right-hand side of (2.19) is convergent on some region, the left-hand side is its limit and if we can further show that the left-hand side can also be represented by a Laurent series on that same region, then this proposition will follow by uniqueness of representation. The first claim is addressed by the next lemma.
Lemma 2.2.4 Set \( x_j = e^{\alpha_j} \), for \( 1 \leq j \leq |\Delta| \). Then, for each \( w \) in \( W \) and \( \lambda \) in \( P \), the series

\[
(-1)^{|\Delta_w|} \sum_{k \in \mathbb{Z}^{|\Delta|}} \exp(w\lambda + \sum_{\alpha \in \Delta_w} (k_\alpha + 1)w\alpha - \sum_{\alpha \in \Delta'_w} k_\alpha w\alpha) \quad (2.20)
\]

converges absolutely in the polydisc \( U = \{ x \in \mathbb{C}^n \mid |x_i| < 1 \} \) to the rational function

\[
e^{w\lambda} \prod_{\alpha \in \Delta} (1 - e^{-w\alpha}). \quad (2.21)
\]

Furthermore, in (2.19), the right-hand side converges absolutely to the left-hand side in \( U \).

Proof. For \( j = 1, \ldots, n \), set \( w\alpha_j = w_1 \alpha_1 + \cdots + w_n \alpha_n \), where \( n = |\Delta| \). For a fixed \( j \), since \( \alpha_j > 0 \), we must have either all \( w_1 \leq 0 \) (\( w\alpha_j < 0 \)) or all \( w_1 \geq 0 \) (\( w\alpha_j > 0 \)) for \( i = 1, \ldots, n \). Hence the terms in (2.20) and the denominator in (2.21) involve only non-negative integral powers of \( x_j = e^{-\alpha_j} \) for \( j = 1, \ldots, n \). Furthermore, up to the shift by \( e^{w\lambda} \), series (2.20) is a multigeometric series in the \( x_j \), hence it converges absolutely in the polydisc \( U \) and uniformly in compacta \( K \subset U \) to the rational function

\[
(-1)^{|\Delta_w|} \frac{\exp(w\lambda + \sum_{\alpha \in \Delta_w} w\alpha)}{\prod_{\alpha \in \Delta_w} (1 - e^{w\alpha}) \prod_{\alpha \in \Delta'_w} (1 - e^{-w\alpha})},
\]

which is (2.21). The second statement is a consequence of the absolute convergence since the series in the right-hand side of (2.19) is obtained by summing (2.20) over \( W \) and rearranging terms.

Proof of 2.2.3. From the Lemma, we know that in (2.19) the rational function on the left-hand side is represented in \( U \) by the power series in the right-hand side. On the other hand, we have from Theorem 2.1.6, which will be proved completely independently below, that the same rational function can also represented in \( U \) by the Laurent polynomial on the right-hand side of (2.14). From the uniqueness of expansion, we conclude that series and polynomial are identical, hence the right-hand side of (2.19) is supported on \( P^\lambda \), as required.

Now we proceed to the proof of Theorem 2.1.6. Recall the Weyl Character Formula (WCF for short) [15, §24.3]. Here we will extend it to all (not necessarily dominant) weights.
**Proposition 2.2.5 (WCF)** If $\lambda$ is any weight and $\chi_\lambda$ is a signed character then

$$a_\rho \chi_\lambda = a_{\lambda + \rho}, \quad (2.22)$$

where $a_\mu = \sum_{w \in W} (-1)^{l(w)} e^{w\mu}$.

The expression $a_\rho$ is called the Weyl Denominator. It is well-known that it satisfies $a_\rho = e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha})$.

**Proof.** First observe that $a_\mu$ is $W$-alternating, that is, $w a_\mu = a_{w\mu} = (-1)^{l(w)} a_\mu$. If $\lambda$ is dominant, (2.22) is just the classical WCF, so there is nothing to show. Now, if $\lambda + \rho$ is not regular, then $\langle \lambda + \rho, \alpha \rangle = 0$, so $s_\alpha(\lambda + \rho) = \lambda + \rho$ for some root $\alpha$. Therefore, 

$$-a_{\lambda + \rho} = s_\alpha(a_{\lambda + \rho}) = a_{s_\alpha(\lambda + \rho)} = a_{\lambda + \rho},$$

forcing $a_{\lambda + \rho} = 0$. But in this case the left-hand side of (2.22) will be zero by definition, so we are done. Finally, if $\lambda + \rho$ is regular, then $\lambda = w_\lambda(\lambda + \rho) - \rho$ is dominant. Moreover, $a_{\lambda + \rho} = (-1)^{l(w)} a_{\lambda + \rho}$, so by the classical WCF, this is $(-1)^{l(w)} a_{\lambda + \rho} = a_\rho (-1)^{l(w)} \chi_\lambda$ which is equal to $a_\rho \chi_\lambda$. 

We are going to show that (2.6) can be formally represented by a Laurent polynomial. Therefore, since the region $U$ defined above contains no poles, (2.6) can be represented as a rational function by that polynomial on $U$.

**Proof of 2.1.6.** With the above considerations, we just have to carry out a straightforward calculation:

$$a_\rho \sum_{w \in W} w \cdot \frac{e^\lambda}{\prod_{\alpha \in \Delta} (1 - e^{-\alpha})} =$$

$$= \sum_{w \in W} (-1)^{l(w)} w \cdot e^{\lambda + \rho} \prod_{\alpha \in \Delta'} (1 - e^{-\alpha})$$

$$= \sum_{w \in W} (-1)^{l(w)} w \cdot e^{\lambda + \rho} \left( 1 + \sum_{\Phi} c_\Phi e^{-(\Phi)} \right)$$

$$= a_{\lambda + \rho} + \sum_{\Phi} c_\Phi \sum_{w \in W} (-1)^{l(w)} e^{w(\lambda - \langle \Phi \rangle + \rho)}$$

$$= a_{\lambda + \rho} + \sum_{\Phi} c_\Phi a_{\lambda - \langle \Phi \rangle + \rho}$$

$$= a_{\lambda + \rho} + a_\rho \sum_{\Phi} c_\Phi \chi_{\lambda - \langle \Phi \rangle},$$

where the summations on the right-hand side are over all nonempty subsets $\Phi$ of $\Delta'$ and the last step is due to our version of the WCF. \[\blacksquare\]
Finally, we will prove Corollary 2.1.4. For complex variables \( s, z_1, \ldots, z_n \), consider the function

\[
F(s, z_1, \ldots, z_n) = \prod_{i=1}^{n} \frac{s z_i}{1 - e^{-s z_i}}.
\]

It is well-known that this is analytic in a neighborhood of 0 in \( \mathbb{C}^{n+1} \) and hence has a power series expansion of the form

\[
F(s, z_1, \ldots, z_n) = \sum_{k=0}^{\infty} s^k T_k(z_1, \ldots, z_n),
\]

where each \( T_k \) is a homogeneous polynomial of degree \( k \) known as the \( k \)-th Todd polynomial in \( z_1, \ldots, z_n \).

Interpreting \( e^{\mu} \) as the complex function \( t \to e^{t \langle \mu, \rho \rangle} \), we can rewrite formula (2.13) as

\[
\sum_{w \in W} \frac{e^{t \langle w\lambda, \rho \rangle}}{\prod_{i=1}^{n} (1 - e^{-t \langle w\alpha_i, \rho \rangle})} = \sum_{\mu \in P} e^{t \langle \mu, \rho \rangle}. \tag{2.23}
\]

Each term on the left-hand side is a meromorphic function with a pole at \( t = 0 \), so it can be expanded as a Laurent series:

\[
\sum_{w \in W} \frac{e^{t \langle w\lambda, \rho \rangle}}{\prod_{i=1}^{n} t \langle w\alpha_i, \rho \rangle} \sum_{k=0}^{\infty} T_k(\langle w\alpha_1, \rho \rangle, \ldots, \langle w\alpha_n \rangle) t^k = \sum_{\mu \in P} e^{t \langle \mu, \rho \rangle}. \tag{2.24}
\]

By comparing the zero-th order term on each side we obtain (2.10), which proves Corollary 2.1.4.

### 2.3 Weights of \( V^\rho \)

The lemmas in this section are standard and are reproduced here for convenience. We use them to prove Theorem 2.1.1 at the end.

**Lemma 2.3.1 (Kostant)** Let \( \mu \) be a weight in \( P \). Then \( \mu \) belongs to \( P^\rho \) if and only if \( \mu = \rho - \langle \Phi \rangle \) for some subset \( \Phi \) of \( R^+ \). Moreover, the multiplicity \( m_\rho(\mu) \) is equal to the number of such subsets.

**Proof.** This is Lemma 5.9 in [25].

For each \( w \) in the Weyl group \( W \) define \( \Phi_w = wR^- \cap R^+ \), namely the set of all positive roots that are sent to negative roots by \( w^{-1} \) and let \( \Phi'_w \) be its complement in \( R^+ \).
Lemma 2.3.2 Let $w_0$ be the longest element of $W$. Then for each $w$ in $W$ we have $\Phi'_w = \Phi_{ww_0}$ and

$$w\rho = \frac{1}{2}(\Phi'_w) - \frac{1}{2}(\Phi_w).$$

Furthermore, each set $\Phi_w$ and $\Phi'_w$ is either empty or contain a simple root.

Proof. We have $\Phi'_w = R^+ \setminus (wR^- \cap R^+) = wR^+ \cap R^+$. Also, $w_0$ exchanges $R^+$ with $R^-$, so $\Phi'_w = ww_0R^- \cap R^+ = \Phi_{ww_0}$. Next we compute

$$wp = \frac{1}{2}(wR^+) = \frac{1}{2}((wR^+ \cap R^+) \dot{\cup} (wR^+ \cap R^-)) = \frac{1}{2}(ww_0R^- \cap R^+) - \frac{1}{2}(wR^- \cap R^+) = \frac{1}{2}(\Phi_{ww_0}) - \frac{1}{2}(\Phi_w),$$

which shows the second statement. Now suppose $\Phi_w$ contains $\Delta$. Then, in particular, the positive system $wR^-$ contains the simple system $\Delta$ forcing $wR^- = R^+$. Thus $w = w_0$ and $\Phi'_w$ is empty. Repeating the same argument with $ww_0$ instead of $w$ we see that if $\Phi'_w$ contains $\Delta$ then $\Phi_w$ is empty.

Lemma 2.3.3 If $\Phi$ is a subset of $R^+$ then

$$\langle \rho - \langle \Phi \rangle, \rho - \langle \Phi \rangle \rangle \leq \langle \rho, \rho \rangle,$$

and equality holds if and only if $\Phi = \Phi_w$ for some $w$ in $W$.

Proof. The inequality is obvious from Lemma 2.3.1. It is also clear that the equality will hold if and only if there exists $w$ in $W$ such that $wp = \rho - \langle \Phi \rangle$. In that case, from Lemma 2.3.1 we must have $\Phi = \Phi_w$.

Proof of 2.1.1. After Theorems 2.1.5 and 2.1.6, all that remains to be done is to show that (2.5) is indeed a recursion. For this purpose, it is enough to show for $\lambda$ dominant and $\Phi$ a nonempty subset of $\Delta'$ that $\overline{\lambda - \langle \Phi \rangle} = w_\lambda(\lambda - \langle \Phi \rangle + \rho) - \rho < \lambda$.

Let $\mu = \overline{\lambda - \langle \Phi \rangle}$. Since $\lambda - \langle \Phi \rangle + \rho \leq \lambda + \rho$, then

$$\mu + \rho = w_\lambda(\lambda - \langle \Phi \rangle + \rho) \leq \lambda + \rho,$$
so \( \mu \leq \lambda \). If we had \( \mu = \lambda \), then we would also have
\[
\langle \lambda - \langle \Phi \rangle + \rho, \lambda - \langle \Phi \rangle + \rho \rangle = \langle \lambda + \rho, \lambda + \rho \rangle.
\]
But then Lemma 2.3.3 would imply that
\[
2\langle \lambda, \langle \Phi \rangle \rangle = \langle \rho - \langle \Phi \rangle, \rho - \langle \Phi \rangle \rangle - \langle \rho, \rho \rangle \leq 0,
\]
with the equality holding if and only if \( \Phi = \Phi_w \) for some \( w \). From Lemma 2.3.2, this is impossible because \( \lambda \) is dominant, \( \Phi \) is nonempty and it contains no simple root. Therefore \( \mu < \lambda \), as required.

Finally we observe that \( \mu \) will fail to be dominant only if \( \lambda - \langle \Phi \rangle + \rho \) is not regular, but in this case \( \chi_\mu \) will be zero, so we may assume that \( \mu \) is dominant. We conclude the recursion must stop after finitely many steps due to the fact that there are only finitely many dominant weights \( \mu \leq \lambda \).

### 2.4 Convex Geometry

Since our results are of an immediate representation-theoretic significance, we have resorted only to elementary methods of Lie Theory to prove them. However, we must acknowledge to the fact that Theorem 2.1.5 can also be derived, slightly indirectly, from results in Convex Geometry due to Brion [5] and Brion & Vergne [6]. The purpose of this section is to explain how this can be done, at the expense of introducing a considerable amount of geometrical notions.

Let \( E \) be an euclidean space of dimension \( n \) with form \( \langle \cdot, \cdot \rangle \) and fix a lattice \( Q \) in \( E \) (e.g. the root lattice) with basis \( \alpha_1, \ldots, \alpha_n \), i.e. a spanning set for \( Q \) over \( \mathbb{Z} \). If \( \alpha'_1, \ldots, \alpha'_n \) is a set of vectors satisfying \( \langle \alpha_i, \alpha'_j \rangle = 1 \) if \( i + j = n + 1 \) and zero otherwise, then \( \alpha'_1, \ldots, \alpha'_n \) is called the dual basis of \( \alpha_1, \ldots, \alpha_n \) and the lattice \( Q' \) they span is called the dual lattice of \( Q \). This is also the set \( \{ x \in E \mid \langle x, y \rangle \in \mathbb{Z}, \text{ for all } y \in Q \} \).

A cone is a subset \( C \) in \( E \) such that \( 0 \in C \) and for all \( x \in C \), \( tx \in C \) for all real \( t \geq 0 \). \( C \) is convex if for any pair of points \( x, y \) in \( C \), \( tx + sy \) is also in \( C \) for all real \( s, t \geq 0 \). Given any points \( e_1, \ldots, e_m \) in \( E \), the set
\[
C = \{ a_1 e_1 + \cdots + a_m e_m \mid a_i \geq 0 \text{ for } i = 1, \ldots, m \}
\]
of all conic combinations of these points is a convex cone. We say that $e_1, \ldots, e_m$ are generators of $C$. If the generators of $C$ belong to the lattice $Q$ then we say that $C$ is integral with respect to $Q$. The generators for $C$ are called shortest if for all $j = 1, \ldots, m$, $te_j \in Q$ implies $t \in \mathbb{Z}$. A cone is said to be simplicial if its number of edges equals the dimension of the linear subspace it spans. The set

$$C' = \{x \in E \mid \langle x, y \rangle \geq 0, \text{ for all } y \in C\}$$

is a cone in $E$ called the dual cone to $C$.

A polytope $P$ is the convex hull of a finite set of points in $E$. It is well-known that every polytope is a polyhedron, i.e. the intersection of finitely many affine half-spaces of $E$, and every bounded polyhedron is again a polytope. We denote the set of vertices of $P$ by $V(P)$. If $V(P) \subset Q$ then $P$ is called a lattice polytope. A polytope is called simple if exactly $n$ edges go through each vertex. For the remainder of this section we shall assume that $P$ is a convex simple lattice polytope, and hence that all its tangent cones are simplicial.

The tangent cone at a vertex $v$ of $P$ is the set

$$C_v = \{t(x - v) \mid \text{for all } x \in P \text{ and all } t \geq 0\}.$$ 

Since $P$ is a simple lattice polytope, we can find a set of shortest generators $v_1, \ldots, v_n$ for $C_v$ that is also a basis for $E$. The dual basis $v'_1, \ldots, v'_n$ of $v_1, \ldots, v_n$, i.e. the vectors $v'_j$ satisfying $\langle v_i, v'_j \rangle = 1$ if $i + j = n + 1$ and zero otherwise, is a set of generators for the dual cone $\sigma_v$ of $C_v$.

Since $P$ is a lattice polytope, $v'_1, \ldots, v'_n$ span a sublattice $U_v$ of $Q'$. Set $G_v = Q'/U_v$. This is an abelian group and its characters $\chi_g(x)$ for $g \in G_v$ are equal to $e^{2\pi i \langle x, y \rangle}$ where $y$ is a representative of $g$ in $Q'$. Then we have the following result of Brion [6, Proposition 3.9]:

**Theorem 2.4.1** If $P$ is a convex simple lattice polytope in $E$, then as generalized functions,

$$\sum_{m \in P \cap Q} e^m = \sum_{v \in V(P)} \frac{e^v}{|G_v|} \sum_{g \in G_v} \frac{1}{\prod_{i=1}^n (1 - \chi_g(v_i) e^{v_i})}.$$  

(2.25)
From this we can now we derive Theorem 2.1.5 as follows. Let $\lambda$ be a dominant weight in the weight lattice $P$. Then the convex hull $P$ of $P^\lambda$ is a convex simple lattice polytope relative to the weight lattice $P$, with vertices $V(P) = W \cdot \lambda$. Furthermore, if $0 \in P^\lambda$, then $P$ is a convex simple lattice polytope relative to the root lattice $Q$. If $\lambda$ is regular, then $V(P)$ is a full set of vertices in the sense that $|V(P)| = |W|$. Therefore we have $G_v = \{0\}$, and Brion’s identity (2.25) becomes precisely (2.13).

The same holds even if $0 \not\in P^\lambda$, since (2.25) is clearly invariant by translations. Now the question arises as to what happens when $\lambda$ is not regular, since in this case the tangent cone $C_v$ at a vertex $v = w \cdot \lambda$ is no longer generated by $-w \cdot \alpha_1, \ldots, -w \cdot \alpha_n$. Nevertheless, if $C_{w',\alpha}$ is the cone generated by $-w' \cdot \alpha_1, \ldots, -w' \cdot \alpha_n$, then $C_v$ is the intersection of all cones $C_{w',\alpha}$ for $w' \in \text{Stab}_W \lambda$. The union of any two distinct cones $C_{w_1,\alpha}$ and $C_{w_2,\alpha}$ can be readily seen to contain a line, thus an application of the inclusion-exclusion formula as in Barvinok [2] shows that

$$\sum_{w' \in \text{Stab}_W \lambda} e^{w'w \cdot \lambda} \prod_{\alpha \in \Delta} (1 - e^{-w'w \cdot \alpha}) = \prod_{\beta \in \mathcal{B}} (1 - e^{-w \cdot \beta}),$$  \hspace{1cm} (2.26)

where $\mathcal{B}$ denotes the set of generators of $C_{\lambda}$. Therefore (2.13) is the same as

$$\sum_{w \in W / \text{Stab}_W \lambda} w \cdot \prod_{\beta \in \mathcal{B}} (1 - e^{-\beta}) = \Theta_{\lambda},$$  \hspace{1cm} (2.27)

which is precisely Brion’s formula for the polytope whose vertices are $w \cdot \lambda$ for $w \in W / \text{Stab}_W \lambda$.

### 2.5 Some remarks

Since the scalar product is $W$-invariant, we can rewrite expression (2.10) as

$$\sum_{w \in W} \frac{1}{\prod_{i=1}^n \langle w \rho, \alpha_i \rangle} \sum_{j=0}^n \frac{\langle w \rho, \lambda \rangle^j}{j!} T_{n-j}(\langle w \rho, \alpha_1 \rangle, \ldots, \langle w \rho, \alpha_n \rangle)$$  \hspace{1cm} (2.28)

which becomes a sum on the orbit of the regular weight $\rho$. This formula may be easier to compute because it is often easier to generate the orbit of a weight under the action of a group than to generate group elements themselves. It would be interesting to find a representation-theoretic interpretation of this formula.
Now that we have a formula for the total number of weights $|P^\lambda|$, it seems natural to ask for a formula just for the number $|P^\lambda_+|$ of dominant integral weights that are smaller than or equal to $\lambda$. The set $P^\lambda_+$ is the intersection of the cone generated from the origin by the fundamental weights with the cone generated from $\lambda$ by $-\alpha_1, \ldots, -\alpha_n$, that is, the polytope with facets given by the inequalities $\langle \mu, \omega_i \rangle \geq 0$ and $\langle \mu - \lambda, \omega_i \rangle \leq 0$ for $i = 1, \ldots, n$. Although it seems that we could simply apply Brion’s formula to that polytope, it turns out that for certain vertices $v$ the corresponding group $G_v$ can be very large. For example, for the representation $V^\rho$ of $E_8$ one finds a group corresponding to a certain vertex containing as many as 823543 elements. The problem here is that the tangent cone at such vertices is not primitive, that is, its unit half-open parallelepiped may contain more than one point. Barvinok and Pommersheim [3] suggested a polynomial-time algorithm to break up non-primitive cones into unions of primitive ones which could then be combined using the inclusion-exclusion principle in the calculation of the number of lattice points. Even though this algorithm was recently implemented on a computer, it is not clear whether one can obtain closed formula. As far as we know, an exact formula for $P^\lambda_+$ is still unknown.

It is also interesting to observe that expression (2.10), seen as a polynomial $p_\rho$ in the entries of $\lambda$ with respect to the weight basis, shows only nonnegative coefficients for degrees less than or equal to $n$ for all Lie algebras studied, namely for $A_2$, $A_3$, $A_4$, $B_2$, $B_3$, $D_4$, $E_6$, $F_4$, and $G_2$. We do not have a reasonable explanation for this occurrence.
Chapter 3

Degree 1 Peri formula for nonsymmetric Jack polynomials

Dating back to the 19th century, the classical Pieri rule is a combinatorial description of the product of a Schur polynomial by an elementary symmetric polynomial. Jack polynomials are generalizations of Schur polynomials and a Pieri rule for them was obtained by Stanley in 1989. Nonsymmetric analogs of Jack polynomials were introduced by Heckman and Opdam following the work of Cherednik. The Pieri rule for these is not yet known. Here we give a complete answer for the product with a degree 1 nonsymmetric factor.

3.1 Introduction

The nonsymmetric Jack polynomials $E_\eta(x)$ are a family of multivariate polynomials in $x = (x_1, \ldots, x_n)$ with coefficients in the field $\mathbb{Q}(\alpha)$ which are indexed by compositions, that is, by $n$-tuples of non-negative integers. They can be defined as the unique polynomials of the form

$$E_\eta(x) = x^\eta + \sum_{\mu < \eta} u_{\eta\mu}(\alpha)x^\mu$$

that are simultaneous eigenfunctions of the family of commuting differential-reflection operators of Cherednik [8]

$$\xi_i = \alpha x_i \frac{\partial}{\partial x_i} + \sum_{k < i} \frac{x_i}{x_i - x_k}(1 - s_{ik}) + \sum_{k > i} \frac{x_k}{x_i - x_k}(1 - s_{ik}) + 1 - i,$$

where $s_{ik}$ is the transposition permuting $x_i$ and $x_k$ and the monomial $x^\eta$ is equal to $x_1^{\eta_1} \cdots x_n^{\eta_n}$. The ordering $<$ is defined as follows. For a composition $\eta$, let $\eta^+$ denote the corresponding partition, that is, the (minimal) reordering of the entries in $\eta$ in non-increasing order. The degree $|\eta|$ of a composition $\eta$ is defined as the sum of all of its entries. This is often called the weight or modulus of $\eta$ but we prefer to call it degree
since $E_\eta(x)$ is homogeneous of degree $|\eta|$. Then $\mu < \eta$ iff $|\mu| = |\eta|$ and either $\mu^+ < \eta^+$, or in case $\mu^+ = \eta^+$, $\mu < \eta$, where the ordering $<$ is the usual dominance ordering on compositions of same degree defined by $\mu < \eta$ iff $\mu \neq \eta$ and $\sum_{i=1}^{k} \mu_i \leq \sum_{i=1}^{k} \eta_i$ for all $k = 1, \ldots, n$. The $E_\eta(x)$ satisfy the eigenvalue equation

$$
\xi_i E_\eta(x) = \pi_i E_\eta(x),
$$

(3.3)

for each $i = 1, \ldots, n$, where the eigenvalue $\pi_i$ is given by

$$
\pi_i = \eta_i \alpha - \#\{k < i \mid \eta_k \geq \eta_i\} - \#\{k > i \mid \eta_k > \eta_i\}. \tag{3.4}
$$

In general, the coefficients of $E_\eta(x)$ are rational functions of $\alpha$. However, Knop and Sahi [22] have shown that a normalization of that polynomial, namely $F_\eta(x) = d_\eta E_\eta(x)$, is integral in the sense that its coefficients are polynomials in $\mathbb{Z}_+[\alpha]$. The constant $d_\eta$, called the upper hook-polynomial of $\eta$, is computed as follows. For a box $s = (i, j)$ in the diagram of $\eta$, that is, the set $\{(i, j) \mid 1 \leq i \leq n, \text{ and } 1 \leq j \leq \eta_i\}$, we define the arm and the leg-lengths of $s$ by

$$
a_\eta(s) = \eta_i - j,
$$

$$
l_\eta(s) = \#\{k < i \mid j \leq \eta_k + 1 \leq \eta_i\} + \#\{k > i \mid j \leq \eta_k \leq \eta_i\}. \tag{3.5}
$$

We also define the upper hook and the lower hook-lengths of $s$ by

$$
d_\eta(s) = (a_\eta(s) + 1)\alpha + l_\eta(s) + 1, \quad d'_\eta(s) = (a_\eta(s) + 1)\alpha + l_\eta(s). \tag{3.6}
$$

Then the upper hook and the lower hook-polynomials are given by

$$
d_\eta = \prod_{s \in \eta} d_\eta(s), \quad d'_\eta = \prod_{s \in \eta} d'_\eta(s). \tag{3.7}
$$

For later use, we shall also define the coarm and coleg-lengths of $s$ by

$$
a'_\eta(s) = j - 1, \quad l'_\eta(s) = \#\{k < i \mid \eta_k \geq \eta_i\} + \#\{k > i \mid \eta_k > \eta_i\}. \tag{3.8}
$$

The coleg-length of $s$ is just the index of the image of row $i$ through the minimal permutation sending $\eta$ to $\eta^+$. If we define the content of $s$ by

$$
c_\eta(s) = (a'_\eta(s) + 1)\alpha - l'_\eta(s), \tag{3.9}
$$
then the eigenvalue $\eta_i$ can be interpreted combinatorially as the content of the rightmost box of row $i$ of $\eta$. If we apply these definitions to partitions then, with the exception of $d_\eta$, they become just the corresponding ones that are defined only for partitions.

We shall express our combinatorial formula in terms of the dual basis of the $F_\eta$ with respect to a scalar product introduced by Sahi [34]. Define a scalar product in $\mathbb{Q}(\alpha)[x_1, \ldots, x_n]$ by setting $\langle x_\eta, q_\nu \rangle = \delta_\eta \nu$, where the polynomials $q_\nu(x)$ are defined by the following expansion of the nonsymmetric Cauchy kernel

$$\prod_{i=1}^n \frac{1}{(1 - x_i y_i)} \prod_{i,j=1}^n \frac{1}{(1 - x_i y_j)^{1/\alpha}} = \sum_\eta q_\eta(x) y^\nu.$$  

(3.10)

With respect to this product, the $F_\eta$ have the following remarkable properties.

**Theorem 3.1.1 ([34])** The following holds for nonsymmetric Jack polynomials:

1. The polynomials $F_\eta(x)$ are mutually orthogonal with respect to $\langle \cdot, \cdot \rangle$.

2. $f_\eta := \langle F_\eta, F_\eta \rangle = d_\eta d'_\eta$.

Hence the dual of $F_\eta(x)$ with respect to this scalar product is simply the polynomial

$$F_\eta^*(x) = \frac{1}{d_\eta d'_\eta} F_\eta(x).$$  

(3.11)

Next we introduce a notation that turns out to be very handy in stating and proving our main theorems, which also seems to be a good organizing principle when working with compositions.

**Definition. 3.1.2 (Cords)** Let $\eta$ be a composition. The (open) cord of $\eta$ of width $\rho$ from $i$ to $j$ is the subset of $\{1, \ldots, n\}$ defined by

$$C_\rho^\eta(i, j) = \{i < k < j \mid \eta_k = \rho\}.$$  

(3.12)

Similarly, using the appropriate inequalities, we define the cords $C_\rho^\eta[i, j]$, $C_\rho^\eta(i, j]$ and $C_\rho^\eta[i, j)$, depending on whether or not the endpoints are to be considered in the cord. We refer to the cord $C_\rho^\eta = C_\rho^\eta[1, n]$ simply as the cord of $\eta$ of width $\rho$. The distinguished cords $C^\eta = C^\eta_{\eta_1}$ and $C^{\eta_1} = C^{\eta_1}_{\eta_1}[1, k]$, where $k = \max\{i \mid \eta_j = \eta_1\text{ for all } j \leq i\}$, are called
the principal cord and the upper principal cord respectively. For two compositions \( \eta \) and \( \lambda \), let \( i \) and \( j \) be respectively the first and last indices such that \( \eta_i \neq \lambda_i \) and suppose \( \eta_i + 1 = \lambda_j \). Then whenever a cord of \( \eta \) has width equal to \( \eta_i + 1 \), we say that the cord is critical relative to \( \lambda \) and we denote it by \( C_{\eta \lambda} \).

**Example 3.1.3** Figure 3.1 shows the principal cord \( C_{\eta} = \{1, 2, 5\} \) for the composition \( \eta = (223121) \). The upper principal cord is \( C_{\eta u} = \{1, 2\} \).

Throughout this work we assume that the elements in a subset of \( \{1, \ldots, n\} \) are naturally ordered.

Following Knop [20], Marshall [29] introduces a partial ordering \( \prec' \) on compositions defined by \( \nu \prec' \eta \) iff there exists a permutation \( \pi \) such that \( \nu_i < \eta_{\pi(i)} \) for \( i < \pi(i) \) and \( \nu_i \leq \eta_{\pi(i)} \) for \( i \geq \pi(i) \). This has the property that for two partitions \( \lambda \) and \( \mu \), \( \lambda \prec' \mu \) is equivalent to \( \lambda \subseteq \mu \) (inclusion of diagrams), but for compositions, although \( \lambda \subseteq \mu \) implies \( \lambda \prec' \mu \), the converse is in general not true. The minimal elements over \( \eta \) with respect to this partial order play a central role in the derivation of the Pieri rule for the nonsymmetric Jack polynomials. These elements can be characterized as the elements of the set

\[
J_\eta = \{ \nu \mid \eta \prec' \nu \prec' \eta + (1^n), |\nu| = |\eta| + 1 \}, \tag{3.13}
\]

where \( (1^n) \) is the composition whose entries are all equal to 1’s. There is a more explicit characterization given by the next definition.

**Definition. 3.1.4** ([20]) Let \( L = \{j_1, \ldots, j_L\} \) be a subset of \( \{1, \ldots, n\} \). Let \( c_L(\eta) \) be the composition \( \mu \) defined by
In other words, \( c_L(\eta) \) is obtained from \( \eta \) by adding one to \( \eta_{j_1} \) and then rotating the entries \( j_1, \ldots, j_\ell \) circularly to the left once.

This makes sense for it is immediately obvious that \( \eta \prec' c_L(\eta) \), and it can be shown [20] that if \( \nu \prec' \mu \) then there exists an \( L \) such that \( c_L(\nu) \prec' \mu \). This, however, can occur for more than one different set \( L \). We can avoid this with the notion of a maximal subset with respect to \( \eta \).

**Definition 3.1.5 ([20])** We say a subset \( L = \{j_1, \ldots, j_\ell\} \) of \( \{1, \ldots, n\} \) is maximal with respect to \( \eta \) if it satisfies:

(a) \( \eta_i \neq \eta_{j_1} \) for \( 1 \leq i < j_1 \),

(b) \( \eta_i \neq \eta_{j_t} \) for \( 1 < t \leq \ell \) and \( j_{t-1} < i < j_t \),

(c) \( \eta_i \neq \eta_{j_1} + 1 \) for \( j_\ell < i \leq n \).

**Proposition 3.1.6** If \( L \) is maximal with respect to \( \eta \) and \( c_L(\eta) = c_M(\eta) \) then \( M \subset L \). Therefore, given \( \eta \) and \( \lambda = c_L(\eta) \) determines \( L \) uniquely.

**Proof.** Let \( \lambda = c_L(\eta) \), \( L = \{j_1, \ldots, j_\ell\} \), and \( M = \{i_1, \ldots, i_s\} \). If \( S = \{i | \eta_i \neq \lambda_i\} \), then \( \lambda = c_S(\eta) \), \( S \subset L \) and \( S \subset M \), so \( S \subset M \cap L \) is not empty. Suppose \( i = i_p \in M \) with \( \eta_{i_p} = \lambda_{i_p} \) and \( i_{p+1} \in M \cap L \). Then \( \eta_{i_p} = \lambda_{i_p} = \eta_{i_{p+1}} \), which forces \( i_p \in L \) by maximality. Hence, if \( k \) is the last element in \( S \), then \( i \in L \) whenever \( i \in M \) and \( i \leq k \). On the other hand, suppose \( i = i_p \in M \) with \( \eta_{i_p} = \lambda_{i_p} \), and \( i_{p-1} \in M \cap L \) for \( i > k \). Then \( \eta_{i_1} + 1 = \lambda_{i_{p-1}} = \eta_{i_p} \), which again forces \( i_p \in L \) by maximality. Hence \( i \in L \) whenever \( i \in M \) and \( i > k \), as required. \( \square \)
3.2 Elementary properties of the nonsymmetric Jack polynomials

The purpose of this section is to give a brief review of the elementary properties of the nonsymmetric Jack polynomials that are relevant to this work.

In [22] Knop and Sahi give an explicit combinatorial formula for the polynomials $F_\eta(x)$ as follows. A generalized tableau of shape $\eta$ is a filling $T$ of the diagram of $\eta$ with the numbers $1, \ldots, n$. To each tableau $T$ of shape $\eta$ we associate the monomial $x^T$ defined by $\prod_{s \in \eta} x_{T(s)}$. We say $T$ is 0-admissible if it satisfies:

(a) $T(i, j) \neq T(k, j)$ for all $k > i$,

(b) $T(i, j) \neq T(k, j-1)$ for all $k < i$ and $j > 1$,

(c) $T(i, 1) \geq i$, for all $i$.

Condition (c) can be seen as equivalent to condition (b) by considering a fictitious leftmost column where each box from top to bottom is successively labeled $1, 2, \ldots, n$. A box $s = (i, j) \in \eta$ is called 0-critical if $T(i, j) = T(i, j-1)$ if $j > 1$ or $T(i, 1) = i$ if $j = 1$. The hook-polynomial of $T$ is defined by

$$d^0_T = \prod_{s \in \eta, s \text{ 0-critical}} d_\eta(s).$$

(3.14)

$$
\begin{array}{cccc}
1 & 1 & 2 & 2 \\
3 & 3 & 3 & 3 \\
x_1 x_2 x_3 & x_1 x_2 x_3 & x_1 x_2 x_3 & x_2 x_3^2
\end{array}
$$

Figure 3.2: 0-admissible tableaux of shape $\eta = (102)$

With this notation, we have the following result.

**Theorem 3.2.1 ([22])** Let $\eta$ be a composition. Then

$$F_\eta(x) = \sum_{T \text{ 0-admissible}} d^0_T x^T.$$  

(3.15)
Example 3.2.2 Figure 3.2 shows all possible 0-admissible tableaux for $\eta = (102)$. The shaded boxes are the 0-critical ones. The corresponding Jack polynomial is

$$F_{(102)} = (2\alpha + 3)(\alpha + 2)x_1x_2x_3$$

$$+ (\alpha + 1)(2\alpha + 3)(\alpha + 2)x_1x_3^2$$

$$+ (2\alpha + 3)(\alpha + 2)x_2x_3^2$$

(3.16)

3.3 The Pieri rule

The classical Pieri rule is a combinatorial rule describing the product of a Schur polynomial and an elementary symmetric polynomial. Stanley [37] obtained a similar rule for the symmetric Jack polynomials, which generalize the Schur polynomials. The Pieri formula for the nonsymmetric analogs of the Jack polynomials is a combinatorial rule describing the coefficients $g^\lambda_{\nu\eta}(\alpha)$ defined by the formula

$$F_\nu F_\eta = \sum_\lambda g^\lambda_{\nu\eta}(\alpha)F^*_\lambda,$$  

(3.17)

when $\nu \in \{0, 1\}^n$. In general, for an unrestricted $\nu$, it follows from the work of Sahi [34] and Knop-Sahi [21] that $g^\lambda_{\nu\eta}(\alpha)$ is a polynomial in $\alpha$.

Although the nonsymmetric Pieri rule is presently unknown in its full generality, in this section we provide a complete description for the $|\nu| = 1$ case, namely we give a formula for the $g^\lambda_{\varepsilon_k\eta}(\alpha)$, where $\varepsilon_k$ is the $k$-th canonical basis vector of $\mathbb{Z}^n$. For simplicity of notation, we will be writing $g^\lambda_{\eta}(\alpha)$ for $g^\lambda_{\varepsilon_k\eta}(\alpha)$ throughout the rest of this work.

We begin with a complete characterization of those coefficients which are nonzero in (3.17) for $\nu = \varepsilon_k$.

**Theorem 3.3.1 (Main #1) The coefficient $g^\lambda_{\varepsilon_k\eta}(\alpha)$ is nonzero if, and only if, the following conditions hold:**

(a) $\lambda = c_S(\eta)$ where $S = \{i_1, \ldots, i_s\} = \{i | \eta_i \neq \lambda_i\}$,

(b) Either $i_s \geq k$ or $\#\{i \geq k | \eta_i = \eta_1 + 1\} > 0$,

(c) If $\eta_1 = \eta_1 = \cdots = \eta_k$ then $i_1 \leq k$. 


Remark 3.3.2  Condition (a) is equivalent to asking that if $\lambda = c_K(\eta)$ for some subset $K$ of $\{1, \ldots, n\}$, then $S \subset K$. Hence $S$ is the minimal subset so that $\lambda = c_S(\lambda)$. Condition (b) means that either the last difference between $\eta$ and $\lambda$ does not occur below $k$ or the critical cord $C_{\eta,\lambda}[k, n]$ is nonempty. Condition (c) means that if $k$ is in the upper principal cord, and the first difference between $\eta$ and $\lambda$ happens to occur somewhere in the principal cord, then it must not occur below $k$. Figure 3.3 shows $\eta = (1121)$ and the compositions occurring in the decomposition of $F_{\varepsilon_2}F_{(1121)}$. The shaded rows denote the elements in the corresponding set $S$.

![Figure 3.3: Compositions occurring in the expansion of $F_{\varepsilon_2}F_{(1121)}$](image)

Next we give a combinatorial description of the nonzero coefficients. This will require the definition of two auxiliary polynomials in $\alpha$, namely $b_{\eta,\lambda}(\alpha)$ and $b^{(k)}_{\eta,\lambda}(\alpha)$. These are defined by the filling of each box in the diagrams of $\eta$ and $\lambda$ with either 1, -1, an upper hook or a lower hook and then taking the product of all the entries according to the rule to be described below. For each individual box $s$, we denote these fillings by $d^\#_{\eta}(s)$ and $d^\#_{\lambda}(s)$ respectively for $\eta$ and $\lambda$.

**Definition. 3.3.3** Let $L = \{j_1, \ldots, j_\ell\}$ be maximal with respect to $\eta$ and let $1 \leq k \leq n$. Then, for each box $s = (i, j)$ in the indicated diagram, we define:

(a) 

$$
\begin{cases} 
  d^\#_{\eta}(s), & \text{if } i \not\in L, \\
  1, & \text{if } i = j_p \in L, p < \ell, \eta_{j_p} > \eta_{j_p+1}, j = \eta_{j_p+1} + 1, \\
  & \text{or } i = j_\ell, \eta_{j_1} + 1 < \eta_{j_\ell}, j = \eta_{j_1} + 2, \\
  d_{\eta}(s), & \text{otherwise}, 
\end{cases}
$$

(3.18)
(b)  
\[ d_\lambda^\alpha(s) = \begin{cases} 
    d_\lambda(s), & \text{if } i \not\in L, \\
    -1, & \text{if } i = j_p \in L, p > 1, \lambda_{j_p-1} < \lambda_{j_p}, \ j = \lambda_{j_p-1} + 1, \\
    & \text{or } i = j_1, \lambda_{j_1} - 1 < \lambda_{j_1}, \ j = \lambda_{j_1}, \\
    d_\lambda'(s), & \text{otherwise}, 
\end{cases} \]  
(3.19)

(c)  
\[ b_{\eta\lambda}(\alpha) = \left( \prod_{s \in \eta} d_\eta^\alpha(s) \right) \left( \prod_{s \in \lambda} d_\lambda^\alpha(s) \right), \]  
(3.20)

(d)  
\[ b_{\eta\lambda}^{(k)}(\alpha) = b_{\eta\lambda}(\alpha) \times \begin{cases} 
    d_\eta(j_p, \eta_{j_p+1} + 1), & \text{if } k = j_p, p < \ell \text{ and } \eta_{j_p} > \eta_{j_p+1}, \\
    d_\eta(j_\ell, \eta_{j_1} + 2), & \text{if } k = j_\ell, \eta_{j_1} + 1 < \eta_{j_\ell}, \\
    -d_\lambda'(j_p, \lambda_{j_p-1} + 1), & \text{if } k = j_p, p > 1 \text{ and } \lambda_{j_p-1} < \lambda_{j_p}, \\
    -d_\lambda'(j_1, \lambda_{j_\ell}), & \text{if } k = j_1 \text{ and } \lambda_{j_\ell} - 1 < \lambda_{j_1}, \\
    0, & \text{otherwise}. 
\end{cases} \]  
(3.21)

Depending on the value of \( k \), we can think of \( b_{\eta\lambda}^{(k)}(\alpha) \) as if we were restoring a hook in \( b_{\eta\lambda}(\alpha) \) corresponding to a box of \( \eta \) which had previously been filled with a 1 or else restoring a hook in \( b_{\eta\lambda}(\alpha) \) corresponding to a box in \( \lambda \) which had previously been filled with a \(-1\).

**Example 3.3.4** For \( \eta = (01312) \) and \( \lambda = (13211) \), we have \( L = \{1, 2, 3, 5\} \) and
\[ b_{\eta\lambda} = 2 \ (\alpha + 3) \ (3 \alpha + 5) \ (2 \alpha + 3)^2 \ (\alpha + 1)^2 \ (\alpha + 2)^2 \ (3 \alpha + 4) \ \alpha^2, \]
which is equal the product of all the entries in the diagrams of figure 3.4, while
\[ b_{\eta\lambda}^{(1)} = -2 \ (\alpha + 3) \ (3 \alpha + 5) \ (2 \alpha + 3)^2 \ (\alpha + 1)^2 \ (\alpha + 2)^3 \ (3 \alpha + 4) \ \alpha^2, \]
which is the product of all the entries in the diagrams of figure 3.5.

With the above definitions, the following result establishes a complete combinatorial description for a nonzero coefficient \( g_{\kappa\eta}^\lambda(\alpha) \).
Theorem 3.3.5 (Main #2) Suppose \( g_{\lambda \eta}^{\lambda}(\alpha) \neq 0 \), \( \lambda = c_L(\eta) \) and \( L = \{j_1, \ldots, j_\ell\} \) is maximal with respect to \( \eta \). Then

\[
g_{\lambda \eta}^{\lambda}(\alpha) = \begin{cases} 
  (\alpha + k)b_{\eta \lambda}^{(k)}(\alpha) + (c_\lambda(j_p) - c_\lambda(j_\ell))b_{\eta \lambda}(\alpha), & \text{if } k = j_p \in L, \\
  (c_\lambda(j_p) - c_\lambda(j_\ell))b_{\eta \lambda}(\alpha), & \text{if } j_p < k < j_{p+1}, 1 \leq p < \ell, \\
  -\alpha b_{\eta \lambda}(\alpha), & \text{if } k < j_1,
\end{cases}
\]

(3.22)

where \( c_\lambda(i) := c_\lambda(i, \lambda_i) \) represents the content of the rightmost box on row \( i \) of \( \lambda \).

Here we need not consider the possibility of \( k > j_\ell \), since in this case we already know from Main Theorem #1 that \( g_{\lambda \eta}^{\lambda}(\alpha) = 0 \).

Next, we shall also give a combinatorial interpretation for the coefficients in the expansion of the product of a single variable with a nonsymmetric Jack polynomial, thus obtaining a combinatorial interpretation for Marshall's formula (to be introduced below).
Theorem 3.3.6 (Main #3) Let $a_{i\eta}^\lambda(\alpha)$ be defined by the expression

$$x_iF_{\eta}(x) = \sum_{\lambda} a_{i\eta}^\lambda(\alpha)F^*_\lambda. \quad (3.23)$$

Then $a_{i\eta}^\lambda(\alpha) \neq 0$ if and only if $\lambda = c_L(\eta)$ for $L$ containing $i$ and maximal with respect to $\eta$. Furthermore, $a_{i\eta}^\lambda(\alpha) = b^{(i)}_{\eta\lambda}(\alpha)$.

This leads to a simple but insightful corollary.

Corollary 3.3.7 Suppose that $g_{k\eta}(\alpha)^\lambda \neq 0$. Then

$$g_{k\eta}^\lambda(\alpha) = \sum_T d^0_T b^{(T(s))}_{\eta\lambda}(\alpha), \quad (3.24)$$

where the summation is over all 0-admissible tableaux of shape $\varepsilon_k$ and $s$ is the only box in $\varepsilon_k$.

Although this seems to be a nice formula, there are many cancellations among its terms. After all such terms collapse, what remains is our formula (3.22).

3.4 Marshall’s Formula

Following Knop and Sahi’s [21] introduction of interpolated nonsymmetric polynomials, Marshall [29] has recently obtained a (non-combinatorial) formula for the product of a nonsymmetric Jack polynomial with a linear function. Further down we shall derive the degree-1 Pieri formula from his by interpreting its coefficients combinatorially. This will require some notation.

For $L = \{j_1, \ldots, j_{\ell}\} \subset \{1, \ldots, n\}$, write

$$A_L(x) = \frac{1}{\alpha^{\ell-1}(x_{j_{\ell}} - x_{j_1})} \prod_{t=1}^{\ell-1} \frac{1}{x_{j_t} - x_{j_{t+1}}} \quad (3.25)$$
\[ \bar{B}_L(x) = \left( \prod_{i=j+1}^{\ell} \frac{x_{j_1} + 1 - x_i - \frac{1}{\alpha}}{x_{j_1} + 1 - x_i} \right) \left( x_{j_1} + 1 + \frac{n-1}{\alpha} \right), \]  
\[ \times \left( \prod_{i=1}^{j-1} \frac{x_{j_1} - x_i - \frac{1}{\alpha}}{x_{j_1} - x_i} \right) \left( \prod_{t=2}^{\ell} \prod_{i=j_{t-1}+1}^{j_t} \frac{x_{j_t} - x_i - \frac{1}{\alpha}}{x_{j_t} - x_i} \right), \]

\[ \tilde{\chi}_L^{(i)}(x) = \begin{cases} 
  x_{j_t} - x_{j_{t+1}}, & \text{if } i = j_t \text{ and } 1 \leq t < \ell, \\
  x_{j_\ell} - x_{j_1} - 1, & \text{if } i = j_\ell, \\
  0, & \text{otherwise.} 
\end{cases} \]  

**Theorem 3.4.1** ([29]) Let \( \eta \) be a composition and \( 1 \leq i \leq n \). Then we have the following decomposition for the product of a single variable with a nonsymmetric Jack Polynomial:

\[ x_i F_\eta(x) = \alpha d_\eta d'_\eta \sum_{L \text{ maximal}} \tilde{\chi}_L^{(i)}(\frac{\eta}{\alpha}) A_L(\frac{\eta}{\alpha}) \bar{B}_L(\frac{\eta}{\alpha}) F^*_c L(\eta)(x), \]  

where the sum is over all subsets \( L \) of \( \{1, \ldots, n\} \) which are maximal with respect to \( \eta \).

Actually, this expression differs slightly from the original formula of Marshall in the sense that he expresses it in terms of the \( E_\eta \) and his summation only runs over those \( L \) containing \( i \). However, we observe that for convenience we have defined \( \tilde{\chi}_L^{(i)} \) to be zero when \( i \notin L \).

Now set \( \eta = \epsilon_k \) in the above combinatorial formula (3.15) for \( F_\eta \), and let \( j \) be the filling of the only box in the tableau \( T \) of shape \( \epsilon_k \). Then \( T \) is 0-admissible if and only if \( j \geq k \). Furthermore, the single box of \( T \) is 0-critical if and only if \( j = k \). Thus

\[ F_{\epsilon_k}(x) = (\alpha + k)x_k + x_{k+1} + \cdots + x_n. \]  

Combining this with (3.28), we have

\[ F_{\epsilon_k}(x) F_\eta(x) = \sum_{L \text{ maximal}} \chi_{k\eta}^{c_L(\eta)} d_\eta d'_\eta A_L(\frac{\eta}{\alpha}) \bar{B}_L(\frac{\eta}{\alpha}) F^*_c L(\eta)(x) \]  

where we define

\[ \chi_{k\eta}^\lambda = \begin{cases} 
  \sum_{i=k}^{n} (\delta_{ik}(\alpha + k - 1) + 1)\alpha \tilde{\chi}_L^{(i)}(\frac{\eta}{\alpha}), & \text{if } \lambda = c_L(\eta), \\
  0, & \text{otherwise.} 
\end{cases} \]
Here $\delta_{ik}$ is the usual Kronecker delta. Since $L$ is uniquely determined by $\eta$ and $\lambda$ (cf. Proposition 3.1.6), comparing (3.17) and (3.30) we have that

$$g^\lambda_{k\eta}(\alpha) = \chi_{k\eta}^\lambda d_\eta d'_\eta A_L(\frac{\eta}{\alpha}) \tilde{B}_L(\frac{\eta}{\alpha}).$$  \hspace{1cm} (3.32)

In what follows, we shall split expression (3.32) for $g^\lambda_{k\eta}(\alpha)$ into two main factors, $\chi_{k\eta}^\lambda$ and $d_\eta d'_\eta A_L(\frac{\eta}{\alpha}) \tilde{B}_L(\frac{\eta}{\alpha})$, so as to obtain combinatorial interpretations for both of them individually.

### 3.5 Proof of the Pieri rule

#### 3.5.1 Proof of the first main theorem

We begin with a characterization of the nonvanishing coefficients in terms of maximal subsets.

**Lemma 3.5.1** Let $\eta$ and $\lambda$ be compositions and let $1 \leq k \leq n$. Then $g^\lambda_{k\eta}(\alpha) \neq 0$ if and only if the following conditions hold:

(a') $\lambda = c_L(\eta)$ for $L = \{j_1, \ldots, j_\ell\}$ maximal with respect to $\eta$.

(b') $j_\ell \geq k$.

(c') If $\eta_k = \eta_{j_1}$, $k = j_p \in L$, $p < \ell$ and $\eta_{j_p} = \eta_{j_p+1}$ then

$$\#\{j_1 < i < j_{p+1} \mid \eta_i = \eta_k\} + 1 \neq k.$$

**Proof.** Since $L$ is maximal with respect to $\eta$, it is easy to see that none of the factors in $A_L(\frac{\eta}{\alpha})$ and $\tilde{B}_L(\frac{\eta}{\alpha})$ can be zero, thus $g^\lambda_{k\eta}(\alpha) \neq 0$ if and only if $\chi_{k\eta}^\lambda \neq 0$.

First suppose that $g^\lambda_{k\eta}(\alpha) \neq 0$. Then it is immediate from from Marshall’s formula (3.28) and definitions (3.31) and (3.27) that (a’) and (b’) hold. To check (c’), assume that
all of the listed conditions hold. Then

$$\chi^\lambda_{k\eta} =$$

$$= (\alpha + k)\alpha \tilde{\chi}^{(k)}_L(\frac{\eta}{\alpha}) + \sum_{i=k+1}^n \alpha \tilde{\chi}^{(i)}_L(\frac{\eta}{\alpha})$$

$$= (\alpha + k)(\overline{\eta}_{jp} - \overline{\eta}_{jp+1}) + \overline{\eta}_{jp+1} - \overline{\eta}_{jp+2} + \cdots + \overline{\eta}_{j_\ell} - \overline{\eta}_{j_1} - \alpha$$

$$= (\eta_{jp} - \eta_{jp+1}) \alpha^2$$

$$+ ((\eta_{jp} - \eta_{jp+1})k + \eta_{jp+1} - \eta_{j_1} - 1 + l'_\eta(j_{p+1}) - l'_\eta(j_p)) \alpha$$

$$+ (l'_\eta(j_{p+1}) - l'_\eta(j_p))k + l'_\eta(j_1) - l'_\eta(j_{p+1})$$

$$= k + l'_\eta(j_1) - l'_\eta(j_{p+1}). \quad (3.33)$$

Since in this case, $$l'_\eta(j_1) - l'_\eta(j_{p+1}) = -\#\{j_1 < i < j_{p+1} | \eta_i = \eta_k\} + 1$$, the conclusion in (c') is true.

Conversely, assume that conditions (a')-(c') hold. There are a few cases to consider.

**Case 1.** Assume that all conditions in (c') are true. Then $$\chi^\lambda_{k\eta}$$ is equal to what was computed in (3.33). Since in this case the conclusion of (c') is also true, it is clear that $$\chi^\lambda_{k\eta} \neq 0$$.

**Case 2.** In the previous case, if just $$\eta_{jp} \neq \eta_{jp+1}$$, then the coefficient of $$\alpha^2$$ in (3.33) is not zero, hence $$\chi^\lambda_{k\eta} \neq 0$$.

**Case 3.** In case 1, if just $$\eta_k \neq \eta_{j_1}$$, then the coefficient of $$\alpha$$ in (3.33) cannot be zero, so once more $$\chi^\lambda_{k\eta} \neq 0$$.

**Case 4.** If $$k = j_\ell$$ then, a direct calculation similar to the above one gives

$$\chi^\lambda_{k\eta} =$$

$$= (\alpha + k)(\overline{\eta}_{j_\ell} - \overline{\eta}_{j_1} - \alpha)$$

$$= (\alpha + k)((\eta_{j_\ell} - \eta_{j_1} - 1)\alpha - l'_\eta(j_\ell) + l'_\eta(j_1)), \quad (3.34)$$

which clearly cannot be zero.

**Case 5.** Finally, assume that $$k \not\in L$$. In this case, we necessarily have $$k < j_\ell$$, so $$L = \{j_1, \ldots, j_p\} \cup \{j_{p+1}, \ldots, j_\ell\}$$ for some $$0 \leq p < \ell$$, with $$j_1, \ldots, j_p < k$$ and $$j_{p+1}, \ldots, j_\ell > k$$. 
Thus
\[
\chi^\lambda_{k\eta} = \\
= \alpha \chi^\lambda_L(j_{p+1}) + \cdots + \alpha \chi^\lambda_L(j_\ell) \\
= (\eta_{j_{p+1}} - \eta_{j_1} - 1) \alpha - l'_{\eta}(j_{p+1}) + l'_{\eta}(j_1),
\]
which once more cannot be zero.

This completes the checking all possible cases, and hence the lemma follows.

With this lemma at hand, we are ready to prove our first main theorem.

Proof of Theorem 3.3.1. First assume that conditions (a)-(c) in the theorem hold. We must show that \(g^\lambda_{k\eta}(\alpha) \neq 0\), which can be done by checking that conditions (a')-(c') in Lemma 3.5.1 hold. Since (a) holds, (a') also holds, because we can always complete the set \(S\) to a maximal set \(L = \{j_1, \ldots, j_\ell\}\) with \(\lambda = c_L(\eta)\). Since \(j_\ell \geq i_s\), in virtue of (b) and maximality, we get \(j_\ell \geq k\), so (b') also holds. To show that (c') holds, we need only assume that \(\eta_k = \eta_{j_1}, k = j_p, p < \ell, \) and \(\eta_{j_p} = \eta_{j_{p+1}}\) and check that the conclusion of (c') is true. However, under these assumptions, at least one of the equalities in (c) must fail, for otherwise \(i_1 \leq k\) and maximality would force \(j_1 = 1, \ldots, k = j_p = p\) and \(\eta_k = \eta_{j_{k+1}}\). But in this case, since \(i_1 + 1 \leq k + 1\), we would have \(\eta_{j_{i_1}} \neq \lambda_{j_{i_1}} = \eta_{j_{i_1+1}} = \eta_{j_{i_1}}\) which is absurd. So, in fact, at least one of those equalities fails, meaning that there are less than \(k - 2\) rows between rows 1 and \(k\) of length equal to \(\eta_k\). Also, by maximality, there are no such rows between rows \(k = j_p\) and \(j_{p+1}\). Therefore \(#\{j_1 < i < j_{p+1} | \eta_i = \eta_k\} + 1 < k\) and (c') holds.

Conversely, suppose that \(g^\lambda_{k\eta}(\alpha) \neq 0\) so (a')-(c') are true. Let \(S = \{i_1, \ldots, i_s\}\) be the set obtained by removing from \(L\) those indices \(i\) for which \(\eta_i = \lambda_i\). Then we still have \(\lambda = c_S(\eta)\) and \(S\) is the set of differences between \(\eta\) and \(\lambda\), so (a) holds. Now if \(i_s < k\), then \(\eta_i = \lambda_i\) for all \(i \geq k\). But \(\lambda_{j_\ell} = \eta_{j_1} + 1, \eta_{j_1} = \eta_{j_1},\) and \(j_\ell \geq k\), so \(\eta_{j_\ell} = \eta_{j_1} + 1\). Thus \(#\{i \geq k | \eta_i = \eta_{i_1} + 1\} > 0\), and (b) also holds. To check (c) we may assume that \(\eta_{i_1} = \eta_1 = \cdots = \eta_k\), so by maximality we also have \(\eta_{j_1} = \eta_k, k = j_k \in L\) and \(j_1 = 1, \ldots, j_k = k\). If we had \(i_1 > k\), since \(i_1 \in L\) and \(\eta_{i_1} = \eta_k\), we would get \(k < \ell\) and \(#\{j_1 < i < j_{k+1} | \eta_i = \eta_k\} + 1 = k\), contradicting (c'). Thus \(i_1 \leq k\) and we are done.
3.5.2 Hook variations

The next task is to prove our second main theorem which gives a combinatorial formula for each nonvanishing coefficient. For that purpose, we shall need a few definitions and lemmas dealing with the variation in the hook length-polynomials of \( \eta \) caused by the adding of a box and permuting the rows.

The set of all cords of \( \eta \) whose widths are less than a certain width and which occur in a given interval plays an important role in the rest of this section. Let \( L = \{ j_1, \ldots, j_\ell \} \) be a subset of \( \{1, \ldots, n\} \). We denote the set of cords between 1 and \( j_1 \) of width less than \( \eta_{j_1} \) by \( \mathcal{A} \), the set of cords between \( j_{t-1} \) and \( j_t \) of width less than \( \eta_{j_t} \) by \( \mathcal{B}(t) \) for \( 1 < t \leq \ell \), and the set of cords between \( j_\ell \) and \( n \) of width less than \( \eta_{j_1} + 1 \) by \( \mathcal{G} \). These ideas are formalized by the following definition.

**Definition. 3.5.2** Let \( L = \{ j_1, \ldots, j_\ell \} \) be a subset of \( \{1, \ldots, n\} \). We define:

(a) \( \mathcal{A} = \{ \eta_i \mid 1 \leq i < j_1, \eta_i < \eta_{j_1} \} \). Also, for \( \rho \in \mathcal{A} \), let \( \mathcal{A}_\rho = \mathcal{C}_\rho(1, j_1) \) and \( \hat{\rho} = \min \mathcal{A}_\rho \).

(b) \( \mathcal{B}(t) = \{ \eta_i \mid j_{t-1} < i < j_t, \eta_i < \eta_{j_t} \} \), if \( 1 < t \leq \ell \). Also, for \( \rho \in \mathcal{B}(t) \), let \( \mathcal{B}_\rho(t) = \mathcal{C}_\rho(j_{t-1}, j_t) \) and \( \hat{\rho} = \min \mathcal{B}_\rho(t) \).

(g) \( \mathcal{G} = \{ \eta_i \mid j_\ell < i \leq n, \eta_i < \eta_{j_1} + 1 \} \). Also, for \( \rho \in \mathcal{G} \), let \( \mathcal{G}_\rho = \mathcal{C}_\rho(j_\ell, n] \) and \( \hat{\rho} = \min \mathcal{G}_\rho \).

At first, the multiple definitions of \( \hat{\rho} \) seem ambiguous but their distinction will be clear from the context. In any case, \( \hat{\rho} \) simply denotes the first entry in the cord of width \( \rho \).

The difference between two contents is another ingredient that appears often both in Marshall’s and in our formulas. The next lemma interprets these differences combinatorially.

**Lemma 3.5.3** For \( 1 \leq i < j \leq n \). Let \( \delta_{i,j,\eta} = \overline{\eta}_i - \overline{\eta}_j \). Then

\[
\delta_{i,j,\eta} = \#\mathcal{C}^\eta_{\min\{\eta_i, \eta_j\}}(i, j) + \begin{cases} 
\frac{d_{\eta}(i, \eta_j + 1), \text{ if } \eta_i > \eta_j,}{}
1, \text{ if } \eta_i = \eta_j, \\
-\frac{d'_{\eta}(j, \eta_i + 1), \text{ if } \eta_i < \eta_j,}{(3.36)}
\end{cases}
\]

where \( \mathcal{C}_\rho(i, j) \) is the open cord of \( \eta \) of width \( \rho \) from \( i \) to \( j \).
Proof. It is enough to check that

\[ l'_\eta(j) - l'_\eta(i) = \#C^\eta_{\min(\eta, \eta)}(i, j) + \begin{cases} 
  l_\eta(i, \eta_j + 1) + 1, & \text{if } \eta_i > \eta_j, \\
  1, & \text{if } \eta_i = \eta_j, \\
  -l_\eta(j, \eta_i + 1), & \text{if } \eta_i < \eta_j.
\end{cases} \tag{3.37} \]

For \( \eta_i < \eta_j \), after performing a minimal permutation sending \( \eta \) to the partition \( \eta^+ \), we see that, in the reordered version, the image of row \( j \) comes on top of the image of row \( i \), and hence the rows which are in-between these images are those rows of length \( \eta_i \) which were previously above row \( i \) in \( \eta \), plus those rows of length \( \eta_j \) which where previously below row \( j \) in \( \eta \) and those rows of length greater than \( \eta_i \) and smaller than \( \eta_j \) which where previously elsewhere in \( \eta \). Therefore,

\[
 l'_\eta(j) - l'_\eta(i) = -\# \{ k < i \mid \eta_k = \eta_i \} \\
 -\# \{ k \mid \eta_i + 1 \leq \eta_k < \eta_j \} \\
 -\# \{ k > j \mid \eta_k = \eta_j \} - 1 \\
 = -\# \{ k < j \mid \eta_i + 1 \leq \eta_k + 1 \leq \eta_j \} \\
 +\# \{ i < k < j \mid \eta_k = \eta_i \} \\
 -\# \{ k > j \mid \eta_i + 1 \leq \eta_k \leq \eta_j \} \\
 = -l_\eta(j, \eta_i + 1) + \#C^\eta_{\min(\eta, \eta)}(i, j). \tag{3.38}
\]

Similarly, if \( \eta_i > \eta_j \), after reordering, the image of row \( i \) comes on top of the image of row \( j \) in \( \eta^+ \), hence the rows which are in-between these images are those of length \( \eta_i \) where previously below row \( i \), plus those rows of length \( \eta_j \) which where previously above row \( j \) and those rows of length greater than \( \eta_j \) and smaller than \( \eta_i \) which where previously
elsewhere. Therefore,

\[
l'_\eta(j) - l'_\eta(i) = -\#\{k > i \mid \eta_k = \eta_i\} \\
- \#\{k \mid \eta_j + 1 \leq \eta_k < \eta_i\} \\
- \#\{k < j \mid \eta_k = \eta_j\} + 1 \\
= -\#\{k > i \mid \eta_j + 1 \leq \eta_k \leq \eta_i\} \\
+ \#\{i < k < j \mid \eta_k = \eta_j\} \\
- \#\{k < i \mid \eta_j + 1 \leq \eta_k + 1 \leq \eta_i\} + 1 \\
= l_\eta(i, \eta_j + 1) + 1 + \#C_{\eta, j}^\eta(i, j).
\] (3.39)

The case \(\eta_i = \eta_j\) is similar to the one above, except that both sets \(\{k > i \mid \eta_j + 1 \leq \eta_k \leq \eta_i\}\) and \(\{k < i \mid \eta_j + 1 \leq \eta_k + 1 \leq \eta_i\}\) are empty, so \(l'_\eta(j) - l'_\eta(i) = 1 + \#C_{\eta, j}^\eta(i, j)\).

In proving our formula, we need to keep track of the exact correspondence between the boxes as we go from \(\eta\) to \(\lambda\) along with their respective hook-length variation. The first step in that direction is understanding how the leg-lengths change. For this we need not assume maximality.

**Lemma 3.5.4** Let \(\lambda = c_L(\eta)\) for some subset \(L = \{j_1, \ldots, j_\ell\}\) of \{1, \ldots, n\}. Then

(a) For \(1 \leq i < j_1\) and \(1 \leq j \leq \eta_i\),

\[
l_\lambda(i, j) = l_\eta(i, j) + \delta(j \leq \eta_{j_1} + 1 \leq \eta_i) - \delta(j \leq \eta_{j_1} \leq \eta_i).
\] (3.40)

(b) For \(1 < t \leq \ell, j_{t-1} < i < j_t\) and \(1 \leq j \leq \eta_i\),

\[
l_\lambda(i, j) = l_\eta(i, j) + \delta(j \leq \eta_{j_t} + 1 \leq \eta_i) - \delta(j \leq \eta_{j_t} \leq \eta_i).
\] (3.41)

(c) For \(j_\ell < i \leq n\) and \(1 \leq j \leq \eta_i\),

\[
l_\lambda(i, j) = l_\eta(i, j) + \delta(j \leq \eta_{j_1} + 2 \leq \eta_i) - \delta(j \leq \eta_{j_1} + 1 \leq \eta_i).
\] (3.42)

(d) For \(1 < t \leq \ell\) and \(1 \leq j \leq \eta_{j_t}\),

\[
l_\lambda(j_{t-1}, j) = l_\eta(j_{t-1}, j)
\] (3.43) \\
+ \#\{j_{t-1} < k < j_t \mid \eta_k = \eta_{j_t}\} \\
- \#\{j_{t-1} < k < j_t \mid j - 1 = \eta_k < \eta_{j_t}\}.
(e) For $1 \leq j \leq \eta_{j_1}$,

$$l_\lambda(j_\ell, j + 1) = l_\eta(j_1, j)$$  
$$+ \#\{k < j_1 \mid j \leq \eta_k = \eta_{j_1}\} - \#\{k < j_1 \mid j - 1 = \eta_k < \eta_{j_1}\}$$  
$$+ \#\{k > j_\ell \mid j + 1 \leq \eta_k = \eta_{j_1} + 1\} - \#\{k > j_\ell \mid j = \eta_k < \eta_{j_1} + 1\}.$$  

(3.44)

In the above identities, $\delta(P)$ is 0 if $P$ is false and 1 if $P$ is true.

**Proof.** The calculation is similar for (a), (b) and (c) so here we just check (a), (d) and (e). For $i < j_1$ and $1 \leq j \leq \eta_i$, we have

$$l_\lambda(i, j) = \#\{k < i \mid j \leq \lambda_k + 1 \leq \lambda_i\} + \#\{k > i \mid j \leq \lambda_k \leq \lambda_i\}$$  
$$= \#\{k < i \mid j \leq \eta_k + 1 \leq \eta_i\} + \#\{k > i, k \not\in L \mid j \leq \eta_k \leq \eta_i\}$$  
$$+ \#\{1 < p \leq \ell \mid j \leq \eta_j \leq \eta_i\} + \delta(j \leq \eta_{j_1} + 1 \leq \eta_i).$$  

(3.45)

For $l_\eta(i, j)$ the calculation is analogous, except that we get a $\delta(j \leq \eta_j \leq \eta_i)$ term at the end, so (a) follows.

For (d), by a direct calculation, we obtain

$$l_\lambda(j_{i-1}, j) = l_\eta(j_1, j)$$  
$$+ \#\{j_{i-1} < k < j_1 \mid \eta_k \leq \eta_{j_1}\} - \#\{j_{i-1} < j < j_1 \mid \eta_k + 1 \leq \eta_{j_1}\}.$$  

which clearly simplifies to the right-hand side of (3.43).

Finally, for (e), a direct calculation shows that

$$l_\lambda(j_\ell, j + 1) = l_\eta(j_1, j)$$  
$$+ \#\{k < j_1 \mid j \leq \eta_k \leq \eta_{j_1}\} - \#\{k < j_1 \mid j \leq \eta_k + 1 \leq \eta_{j_1}\}$$  
$$+ \#\{k > j_\ell \mid j \leq \eta_k - 1 \leq \eta_{j_1}\} - \#\{k > j_\ell \mid j \leq \eta_k \leq \eta_{j_1}\}.$$  

(3.47)

Again, this clearly simplifies to the right-hand side of (3.44).

The next step is to understand how the hook-length polynomials change. Now we assume maximality.
Lemma 3.5.5 Let $\lambda = c_L(\eta)$ for $L = \{j_1, \ldots, j_\ell\}$ maximal with respect to $\eta$. Then

(a) For $1 \leq i < j_1$ and $1 \leq j \leq \eta_i$,

$$d_\lambda(i, j) = d_\eta(i, j) + \delta(\lambda_i \geq \lambda_{j_\ell} \text{ and } j = \lambda_{j_\ell}).$$  \hspace{1cm} (3.48)

(b) For $1 < t \leq \ell$, $j_{t-1} < i < j_t$ and $1 \leq j \leq \eta_i$,

$$d_\lambda(i, j) = d_\eta(i, j) + \delta(\lambda_i > \lambda_{j_{t-1}} \text{ and } j = \lambda_{t-1} + 1).$$  \hspace{1cm} (3.49)

(c) For $j_\ell < i \leq n$ and $1 \leq j \leq \eta_i$,

$$d_\lambda(i, j) = d_\eta(i, j) + \delta(\lambda_i > \lambda_{j_\ell} \text{ and } j = \lambda_{j_\ell}).$$  \hspace{1cm} (3.50)

(d) For $1 < t \leq \ell$ and $1 \leq j \leq \eta_{j_t}$,

$$d_\lambda(j_{t-1}, j) = d_\eta(j_t, j) - \#\{j_{t-1} < k < j_t \mid j - 1 = \eta_k < \eta_{j_t}\}. \hspace{1cm} (3.51)$$

(e) For $1 \leq j \leq \eta_{j_1}$,

$$d_\lambda(j_\ell, j + 1) = d_\eta(j_1, j) \hspace{1cm} (3.52)$$

$$\quad - \#\{k < j_1 \mid j - 1 = \eta_k < \eta_{j_1}\}$$

$$\quad - \#\{k > j_\ell \mid j = \eta_k \leq \eta_{j_1} + 1\}. \hspace{1cm} (3.53)$$

Proof. In each case, the arm length is the same. Considering the maximality, each difference of delta functions in the previous lemma becomes just the respective one which is shown above. For (d) the equality $\eta_k = \eta_{j_1}$ is false, and for (e) the equalities $\eta_k = \eta_{j_1}$ ($i < j_1$) and $\eta_k = \eta_{j_1} + 1$ ($i > j_\ell$) are both false, so the lemma follows.

Now we can finally state what are the changes in the hook polynomial $d_\eta$ as we go from $\eta$ to $\lambda$, namely we obtain a formula for the ratio $d_\lambda/d_\eta$. 
Lemma 3.5.6 Let $\lambda = c_L(\eta)$ for $L = \{j_1, \ldots, j_\ell\}$ maximal with respect to $\eta$. Then

$$d_\lambda/d_\eta =$$

$$= \left( \prod_{i=1}^{j_\ell-1} \frac{d_\lambda(i, \lambda_{j_\ell})}{d_\eta(i, \lambda_{j_\ell})} \right) \left( \prod_{i=0}^{\ell} \frac{d_\lambda(i, \lambda_{j_\ell} + 1)}{d_\eta(i, \lambda_{j_\ell} + 1)} \right) \left( \prod_{t=2}^{\ell} \frac{d_\lambda(i, \lambda_{j_{t-1}} + 1)}{d_\eta(i, \lambda_{j_{t-1}} + 1)} \right)$$

$$\times \left( \prod_{\rho \in \mathcal{A}} \frac{d_\lambda(j_1, \rho + 2)}{d_\eta(j_1, \rho + 1)} \right) \left( \prod_{\rho \in \mathcal{G}} \frac{d_\lambda(j_1, \rho + 1)}{d_\eta(j_1, \rho + 1)} \right) \left( \prod_{t=2}^{\ell} \frac{d_\lambda(j_{t-1}, \rho + 1)}{d_\eta(j_{t-1}, \rho + 1)} \right)$$

$$\times d_\lambda(j_\ell, 1). \quad (3.54)$$

Proof. For the factors on the first row, first observe that if $i < j_1$ and $\lambda_i \geq \lambda_{j_\ell}$, then from (a) of previous lemma, we get $d_\lambda(i, j) = d_\eta(i, j) + \delta(j = \lambda_{j_\ell})$, and if $i < j_1$ and $\lambda_i < \lambda_{j_\ell}$, then from the same equation we get $d_\lambda(i, j) = d_\eta(i, j)$, which explains the leftmost factor.

The remaining factors on that row will follow from similar statements referring to (b) and (c) of the same lemma. This takes care of all quotients coming from rows not listed in $L$.

For the factors on the second row, for $1 \leq j \leq \eta_{j_1}$, we know from (e) of previous lemma that $d_\lambda(j_1, j + 1) = d_\eta(j_1, j)$, unless $\eta_k < \eta_{j_1}$ and $j \in \mathcal{A}$ or $\eta_k < \eta_{j_1} + 1$ and $j + 1 \in \mathcal{G}$. This explains both the leftmost and the central factors on that row. The rightmost factor comes from a similar statement relative to (d) of that same lemma.

The final factor on the third row accounts for the extra box in $\lambda$. \hfill \blacksquare

3.5.3 Proof of the second and third main theorems

We begin with a lemma stating that a nonzero $\chi^\lambda_{k\eta}$ has the general form given by the expression $(\alpha + k)[\text{hook-length}] + [\text{difference of two contents}]$, which provides us with a combinatorial interpretation of that polynomial.
Lemma 3.5.7 For \( \eta \) and \( \lambda \) compositions and \( 1 \leq k \leq n \),

\[
\chi_k^\lambda \begin{cases} 
(\alpha + k)d_\eta(j_p, \eta_{jp} + 1) + c_\lambda(j_p) - c_\lambda(j_\ell), \\
\quad \text{if } k = j_p, 1 \leq p < \ell \text{ and } \eta_{jp} > \eta_{jp+1}, \\
(\alpha + k)d_\eta(j_\ell, \eta_{j_1} + 2), \text{ if } k = j_\ell \text{ and } \eta_{j_1} + 1 < \eta_{j_1}, \\
-(\alpha + k)d_\lambda(j_1, \lambda_{j_1}) + c_\lambda(j_1) - c_\lambda(j_\ell), \text{ if } k = j_1 \text{ and } \lambda_{j_1} - 1 < \lambda_{j_1}, \\
-(\alpha + k)d_\lambda(j_p, \lambda_{jp-1} + 1) + c_\lambda(j_p) - c_\lambda(j_\ell), \\
\quad \text{if } k = j_p, 1 < p \leq \ell \text{ and } \lambda_{j_p} > \lambda_{j_{p-1}}, \\
c_\lambda(j_p) - c_\lambda(j_\ell), \text{ if } j_p < k < j_{p+1} \text{ and } 1 \leq p < \ell, \\
-\alpha, \text{ if } k < j_1, \\
0, \text{ otherwise.}
\end{cases}
\tag{3.55}
\]

Proof. Suppose \( \chi_k^\lambda \neq 0 \). From Lemma 3.5.1, we may assume that \( \lambda = c_L(\eta) \) for \( L = \{j_1, \ldots, j_\ell\} \) a maximal subset of \( \{1, \ldots, n\} \). First suppose that \( k = k_p \in L \) for some \( 1 \leq p < t \). Then, from (3.33) and (3.36), we have

\[
\chi_k^\lambda = (\alpha + k) \begin{cases} 
\quad d_\eta(j_p, \eta_{jp} + 1), \text{ if } \eta_{jp} > \eta_{jp+1}, \\
\quad -d_\eta(j_{p+1}, \eta_{jp} + 1) + \#C_{\eta_{j_{p+1}}}(j_p, \eta_{j_{p+1}}), \\
\quad \text{if } \eta_{jp} < \eta_{jp+1}
\end{cases} + \eta_{jp+1} - \eta_{j_1} - \alpha. \tag{3.56}
\]

Hence if \( p > 1 \) from (3.51),

\[
\chi_k^\lambda = (\alpha + k) \begin{cases} 
\quad d_\eta(j_p, \eta_{jp} + 1), \text{ if } \eta_{jp} > \eta_{jp+1}, \\
\quad -d_\eta(j_{p+1}, \eta_{jp} + 1) + \#C_{\eta_{j_{p+1}}}(j_p, \eta_{j_{p+1}}), \\
\quad \text{if } \eta_{jp} < \eta_{jp+1}
\end{cases} + \eta_{jp+1} - \eta_{j_1} - \alpha, \tag{3.57}
\]

and if \( p = 1 \) from (3.53),

\[
\chi_k^\lambda = (\alpha + k) \begin{cases} 
\quad d_\eta(j_1, \eta_{j_2} + 1), \text{ if } \eta_{j_1} > \eta_{j_2}, \\
\quad -d_\eta(j_\ell, \lambda_{j_\ell} + 1), \text{ if } \lambda_{j_\ell} - 1 < \lambda_{j_\ell}
\end{cases} + \eta_{j_{p+1}} - \eta_{j_1} - \alpha. \tag{3.58}
\]

Now we claim that \( \eta_{j_{p+1}} - \eta_{j_1} - \alpha \) is equal to \( c_\lambda(j_p) - c_\lambda(j_\ell) \). In fact,

\[
\eta_{j_{p+1}} - \eta_{j_1} - \alpha = (\eta_{j_{p+1}} - \eta_{j_1} - 1)\alpha - l_\eta'(j_{p+1}) + l_\eta'(j_1) \\
= (\lambda_{j_p} - \lambda_{j_\ell})\alpha - l_\lambda'(j_p) + l_\lambda'(j_\ell) \\
= c_\lambda(j_p) - c_\lambda(j_\ell). \tag{3.59}
\]
Next, suppose that \( p = \ell \). Then, from (3.34),
\[
(\alpha + k)^{-1} \chi_{kn}^\lambda = (\bar{\eta}_{jt} - \bar{\eta}_{j1} - \alpha) = (\eta_{jt} - \eta_{j1} - 1)\alpha - l'_\eta(jt) - l'_\eta(j1) = -(\lambda_{jt} - \lambda_{j\ell-1})\alpha + l'_\lambda(jt) - l'_\lambda(j\ell-1)
\]
\[
= \begin{cases} 
  d_\eta(jt, \eta_{jt} + 2), & \text{if } \eta_{jt} + 1 > \eta_{jt}, \\
  -d'_\lambda(jt, \lambda_{j\ell-1} + 1), & \text{if } \lambda_{j\ell-1} < \lambda_{jt}.
\end{cases} 
\] (3.60)

Now suppose that \( j_p < k < j_p + 1 \) for \( 1 \leq p < \ell \). Then, from (3.35),
\[
\chi_{kn}^\lambda = (\eta_{j_{p+1}} - \eta_{j1} - 1)\alpha - l'_\eta(j_{p+1}) + l'_\eta(j1) = (\lambda_{jp} - \lambda_{j\ell})\alpha - l'_\lambda(jp) + l'_\lambda(j\ell)
\]
\[
= c_\lambda(j_p) - c_\lambda(j\ell). 
\] (3.61)

Finally, if \( k < j_1 \) then this case is just (3.35) with \( p = 0 \), which completes the proof.  

Next we give a combinatorial interpretation for the factor \( A_L(\bar{\eta}) \) in Marshall’s formula.
For convenience we define \( \bar{\eta}_{ji} = \eta_{ji} \) for \( 1 \leq i \leq \ell \), \( \bar{\eta}_{j\ell+1} = \eta_{j1} + 1 \), and also \( \bar{\lambda}_j = \lambda_{j\ell} - 1 \), and \( \bar{\lambda}_j = \lambda_{ji} \) for \( 1 \leq i \leq \ell \).

**Lemma 3.5.8** Let \( \lambda = c_L(\eta) \) for \( L = \{j_1, \ldots, j_\ell\} \) maximal with respect to \( \eta \). Then
\[
A_L(\bar{\eta}) = \alpha \left( \prod_{t=1}^\ell \frac{1}{d_\eta(jt, \bar{\eta}_{jt+1} + 1)} \right) \left( \prod_{t=1}^\ell \frac{(-1)}{d'_\lambda(jt, \bar{\lambda}_{jt-1} + 1)} \right). 
\] (3.62)

**Proof.** This is a simple calculation. For each \( t = 1, \ldots, \ell - 1 \), from Lemma (3.5.3) in virtue of the maximality of \( L \) we have
\[
\bar{\eta}_{jt} - \bar{\eta}_{jt+1} = \begin{cases} 
  d_\eta(jt, \eta_{jt+1} + 1), & \text{if } \eta_{jt} > \eta_{jt+1}, \\
  1, & \text{if } \eta_{jt} = \eta_{jt+1}, \\
  -d'_\eta(jt+1, \eta_{jt} + 1) + \# \mathcal{C}^\eta(jt, j\ell+1), & \text{if } \eta_{jt} < \eta_{jt+1}.
\end{cases} 
\] (3.63)
Now by (3.51) if \( t > 1 \) this becomes just
\[
\bar{\eta}_j - \bar{\eta}_{j+1} = \begin{cases} 
\eta_j - \eta_{j+1}, & \text{if } \eta_j > \eta_{j+1}, \\
1, & \text{if } \eta_j = \eta_{j+1}, \\
-d_{\lambda}^\prime(j, \lambda_{j-1} + 1), & \text{if } \lambda_{j-1} < \lambda_j.
\end{cases}
\] (3.64)

and if \( t = 1 \),
\[
\bar{\eta}_j - \bar{\eta}_2 = \begin{cases} 
\eta_j - \eta_2, & \text{if } \eta_j > \eta_2, \\
1, & \text{if } \eta_j = \eta_2, \\
-d_{\lambda}^\prime(j, \lambda_j), & \text{if } \lambda_j < \lambda_j + 1.
\end{cases}
\] (3.65)

Furthermore, we compute
\[
\bar{\eta}_{j\ell} - \bar{\eta}_1 - \alpha = \\
= \eta_{j\ell} - \eta_1 - 1\alpha - t'_\eta(j_\ell) + t'_\eta(j_1) \\
= (\lambda_{j\ell-1} - \lambda_j)\alpha - t'_\lambda(j_{\ell-1}) + t'_\lambda(j_\ell) \\
= \begin{cases} 
d_{\eta}(j_\ell, \eta_j + 2), & \text{if } \eta_{j_\ell} > \eta_j + 1, \\
-d_{\lambda}^\prime(j_\ell, \lambda_{j\ell-1} + 1), & \text{if } \lambda_{j\ell-1} < \lambda_{j_\ell}.
\end{cases}
\] (3.66)

Now, we just replace these results in Marshall’s formula for \( A_L(\bar{\eta}_\alpha) \) using the above definitions and the lemma follows.

The next lemma gives a combinatorial description for the factor \( \tilde{B}_L(\bar{\eta}_\alpha) \).

**Lemma 3.5.9** Let \( L = \{ j_1, \ldots, j_{\ell} \} \) be maximal with respect to \( \eta \). Then
\[
\tilde{B}_L(\bar{\eta}_\alpha) = \\
\left( \prod_{\rho = 1}^{\eta_{j_1}} \frac{d_{\lambda}(j_1, \rho)}{d_{\lambda}^\prime(j_1, \rho)} \right) \left( \prod_{t=2}^{\ell} \prod_{\rho = 1}^{\eta_{j_t}} \frac{d_{\lambda}(j_t, \rho)}{d_{\lambda}^\prime(j_t, \rho)} \right) \left( \prod_{t=2}^{\ell} \prod_{\rho = 1}^{\eta_{j_{t-1}}} \frac{d_{\lambda}(j_{t-1}, \rho)}{d_{\lambda}^\prime(j_{t-1}, \rho)} \right) \left( \prod_{t=2}^{\ell} \prod_{\rho = 1}^{\eta_{j_t}} \frac{d_{\lambda}(j_t, \rho + 1)}{d_{\lambda}^\prime(j_t, \rho + 1)} \right) \left( \prod_{t=2}^{\ell} \prod_{\rho = 1}^{\eta_{j_{t-1}}} \frac{d_{\lambda}(j_{t-1}, \rho + 1)}{d_{\lambda}^\prime(j_{t-1}, \rho + 1)} \right) \left( \prod_{t=1}^{\ell} \frac{d_{\lambda}(j_\ell, 1)}{\alpha} \right)
\] (3.67)
Proof. This will be a rather long calculation. First, we give a combinatorial interpretation for the differences appearing in each factor of $\tilde{B}_L(\frac{\pi}{\alpha})$. We can see immediately from Lemma 3.5.3 that

1. For $1 \leq i < j_1$,

$$\eta_{j_1} - \eta_{i} - 1 = \begin{cases} -d_\lambda(i, \lambda_i), & \lambda_i > \lambda_{j_\ell} - 1 \\ d_\lambda'(j_1, \eta_i + 1) - \#C_\eta[i, j_1], & \eta_i < \eta_{j_1} \end{cases}.$$

$$\eta_{j_1} - \eta_{i} - 1 = \begin{cases} -d_\lambda(i, \lambda_i), & \lambda_i > \lambda_{j_{t-1}} - 1 \\ d_\eta'(j_t, \eta_i + 1) - \#C_\eta'[i, j_t], & \eta_i < \eta_{j_t} \end{cases}.$$ (3.69)

3. For $i > j_\ell$,

$$\eta_{j_\ell} - \eta_{i} + \alpha - 1 = \begin{cases} d_\lambda'(j_\ell, \lambda_i + 1) + \#C_\lambda[i, j_\ell, \lambda_i], & \lambda_{j_\ell} > \lambda_i \\ -d_\lambda(i, \lambda_{j_\ell} + 1), & \lambda_{j_\ell} < \lambda_i \end{cases}.$$

Next, we replace these differences in the expression of $\tilde{B}_L(\frac{\pi}{\alpha})$, obtaining successively for its main factors:

$$\prod_{i=1}^{j_1-1} \eta_{j_1} - \eta_{i} = \prod_{i=1}^{j_1-1} \frac{d_\lambda(i, \lambda_i)}{d_\lambda'(i, \lambda_i)} \prod_{i=1}^{j_1-1} \frac{d_\eta'(j_1, \eta_i + 1) - \#C_\eta'[i, j_1]}{d_\eta'(j_1, \eta_i + 1) - \#C_\eta'[i + 1, j_1]}$$

$$\prod_{i=1}^{j_1-1} \frac{d_\lambda(i, \lambda_i)}{d_\lambda'(i, \lambda_i)} \prod_{i=1}^{j_1-1} \frac{d_\eta'(j_1, \eta_i + 1) - \#C_\eta'[i, j_1]}{d_\eta'(j_1, \eta_i + 1) - \#C_\eta'[i + 1, j_1]}.$$ (3.71)
\[
\prod_{t=2}^{\ell} \prod_{i=\lambda_{t-1}+1}^{\lambda_t-1} \frac{\eta_{jt} - \eta_i - 1}{\eta_{jt} - \eta_i} = \\
= \left( \prod_{t=2}^{\ell} \prod_{i=\lambda_{t-1}+1}^{\lambda_t} d_{\lambda}(i, \lambda_{jt-1} + 1) \right) \left( \prod_{t=2}^{\ell} \prod_{i=\lambda_{t-1}+1}^{\lambda_t} d'_{\lambda}(i, \lambda_{jt-1} + 1) \right)
\]

\[
= \left( \prod_{t=2}^{\ell} \prod_{i=\lambda_{t-1}+1}^{\lambda_t-1} d_{\lambda}(i, \lambda_{jt-1} + 1) \right) \left( \prod_{t=\rho \in B(t)} d'_{\eta}(j_\ell, \eta_\rho + 1) \right), \tag{3.72}
\]

and

\[
\prod_{i=\lambda_{\ell-1}+1}^{\lambda_\ell} \frac{\eta_{j_1} - \eta_i + \alpha - 1}{\eta_{j_1} - \eta_i + \alpha} = \\
= \left( \prod_{i=\lambda_{j_\ell}+1}^{\lambda_\ell} d_{\lambda}(i, \lambda_{j_\ell} + 1) \right) \left( \prod_{i=\lambda_{j_\ell}+1}^{\lambda_\ell} d'_{\lambda}(i, \lambda_{j_\ell} + 1) \right)
\]

\[
= \left( \prod_{i=\lambda_{j_\ell}+1}^{\lambda_\ell} d_{\lambda}(i, \lambda_{j_\ell} + 1) \right) \left( \prod_{\rho \in G} \frac{d'_{\lambda}(j_\ell, \lambda_\rho + 1)}{d_{\lambda}(j_\ell, \lambda_\rho + 1) + \#G_\rho} \right). \tag{3.73}
\]

When taking the product of these factors, some cancellations may occur between the main factors (3.71) and (3.73). We claim that for any nonnegative integer \( \rho \) (which may be in \( \mathcal{A} \) or not, and also for \( \rho + 1 \) which may be in \( \mathcal{G} \) or not),

\[
d'_{\eta}(j_{1}, \eta_\rho + 1) - \#A_\rho = d'_{\lambda}(j_\ell, \lambda_{\rho+1} + 1) + \#G_{\rho+1}. \tag{3.74}
\]

In fact, an easy calculation shows that both sides in this identity are equal to

\[(\eta_{j_1} - \rho)\alpha + \#\{1 \leq k \leq n \mid \rho + 1 \leq \eta_k \leq \eta_{j_1}\}.
\]

It is slightly more useful to re-express it as

\[
d'_{\eta}(j_{1}, \rho + 1) - \#A_\rho = d'_{\lambda}(j_\ell, \rho + 2) + \#G_{\rho+1}. \tag{3.75}
\]

Now from equations (3.71)-(3.73) we can collect all the factors appearing in the first row of (3.67), so we only need to deal with those factors which contribute to the second
line of (3.67). By collecting such factors, and taking into account the above cancellations, we arrive at:

\[
\begin{align*}
&\left( \prod_{\rho \in A} \frac{d'_{\lambda}(j_\ell, \rho + 2)}{d'_{\eta}(j_1, \rho + 1)} \right) \left( \prod_{\rho \in A} \frac{d'_{\lambda}(j_\ell, \rho + 2)}{d'_{\eta}(j_1, \rho + 1)} \right) \left( \prod_{\rho \in A} \frac{d'_{\lambda}(j_\ell, \rho + 2)}{d'_{\eta}(j_1, \rho + 1)} \right) \times \\
&\left( \prod_{t=2}^{\ell} \prod_{\rho \in B(t)} \frac{d'_{\lambda}(j_{t-1}, \rho + 1)}{d'_{\eta}(j_t, \rho + 1)} \right) = \\
&\left( \prod_{\rho = 1}^{\eta_{j_1}} \frac{d'_{\lambda}(j_\ell, \rho + 1)}{d'_{\eta}(j_1, \rho)} \right) \left( \prod_{t=2}^{\ell} \prod_{\rho = 1}^{\eta_{j_t}} \frac{d'_{\lambda}(j_{t-1}, \rho + 1)}{d'_{\eta}(j_t, \rho + 1)} \right), \tag{3.76}
\end{align*}
\]

which accounts for the first two factors on the second row of (3.67). Finally, observing that

\[\eta_{j_1} + \alpha + n - 1 = d'_{\lambda}(j_\ell, 1),\]

and multiplying out all the relevant factors, we eventually obtain equation (3.67), as required.

\[\blacksquare\]

We are now ready for the proof of our second main theorem.

**Proof of Theorem 3.3.5.** First observe that by replacing the first row of factors on the right-hand side of (3.54) by the reciprocal of the rest of the right-hand side of (3.54) times \(d'_{\lambda}/d'_{\eta}\) in the corresponding row of the right-hand side of (3.67), we obtain

\[
\bar{B}_L(\eta_\alpha) = \frac{d_{\lambda}}{d_{\eta}} \left( \prod_{i \in L, j=1}^{\lambda_i} \prod_{i \in L, j=1}^{d_{\lambda}(i, j) - 1} \left( \prod_{i \in L, j=1}^{\lambda_i} \prod_{i \in L, j=1}^{d_{\lambda}(i, j)} \right) \times \\
\frac{1}{\alpha} \left( \prod_{i \in L, j=1}^{\eta_i} \prod_{i \in L, j=1}^{d_{\eta}(i, j) - 1} \left( \prod_{i \in L, j=1}^{\eta_i} \prod_{i \in L, j=1}^{d_{\eta}(i, j)} \right) \right) \tag{3.77}
\]

Therefore,

\[
\bar{B}_L(\eta_\alpha)d_{\eta}d'_{\eta} = \left( \prod_{i=1}^{n} \prod_{j=1}^{\eta_i} \prod_{i \in L, j=1}^{d'_{\eta}(i, j)} \left( \prod_{i \in L, j=1}^{\eta_i} d_{\eta}(i, j) \right) \right) \times \\
\left( \prod_{i=1}^{n} \prod_{j=1}^{\lambda_i} \prod_{i \in L, j=1}^{d_{\lambda}(i, j)} \left( \prod_{i \in L, j=1}^{\lambda_i} d_{\lambda}(i, j) \right) \right) \frac{1}{\alpha}. \tag{3.78}
\]
This shows that
\[ b_{\eta\lambda}(\alpha) = A_L(\frac{\eta}{\alpha})B_L(\frac{\eta}{\alpha})d_\eta d'_\eta. \]  
(3.79)

This fact, together with the above formula (3.55) for \( \chi^\lambda_{k\eta} \) shows that \( g^\lambda_{k\eta}(\alpha) \) has the prescribed form.

Finally, we prove our third main theorem.

\textbf{Proof of Theorem 3.3.6.} The first claim is clear from Marshall’s formula (3.28). Assuming \( \lambda = c_L(\eta) \), \( L = \{j_1, \ldots, j_\ell\} \) is maximal with respect to \( \eta \) and \( i = j_p \in L \), then comparing coefficients between (3.28) and (3.23), in view of (3.79) we can see that
\[ a^\lambda_{\eta i}(\alpha) = \alpha\chi^L_{\frac{\pi}{\alpha}}(\eta) b_{\eta\lambda}(\alpha). \]
From equations (3.64), (3.65), and (3.66) above, we also have that
\begin{equation}
\alpha\chi^L_{\frac{\pi}{\alpha}}(\eta) b_{\eta\lambda}(\alpha) = b^{(i)}_{\eta\lambda}(\alpha)
\end{equation}
thus \( \alpha\chi^L_{\frac{\pi}{\alpha}}(\eta) b_{\eta\lambda}(\alpha) = b^{(i)}_{\eta\lambda}(\alpha) \) as required.

\section{3.6 Conclusion}

We have completely discussed the simplest case of the Pieri rule for the nonsymmetric Jack polynomials. A natural question to ask at this point is what the full Pieri rule is. Our present results seem to be offering a good glimpse at what it can be. Actually, judging by some empirical data we have acquired, we believe to be very close to its complete understanding. It is our feeling that we can say a bit more:

\textbf{Conjecture 3.6.1} For \( \nu, \eta \) and \( \lambda \) compositions, the following holds:
\begin{enumerate}
\item \( g^\lambda_{\nu\eta} = 0 \) unless \( \lambda \) can be obtained from \( \eta \) by adding \( |\nu| \) boxes to the rows of \( \eta \), and then rearranging the rows such that the rows with boxes added move downwards or stay stationary, while the rows with no boxes added move upwards or stay stationary\footnote{We are aware of a similar, unpublished statement of Forrester and McAnally [11].}.
\end{enumerate}
More precisely, $g^0 = 0$ unless there is sequence of $m = |\nu|$ subsets $L^{(1)}, \ldots, L^{(m)}$ of \{1, \ldots, n\} where $L^{(i)}$ is maximal with respect to $\lambda^{(i)} = c_{L^{(i)}}(\lambda^{(i-1)})$ for all $1 \leq i \leq m$, $\lambda^{(0)} = \eta$, and $\lambda = \lambda^{(m)}$. Furthermore, for any $i < j$, if $s$ is the last element in $L^{(i)}$ and $t$ is the first element in $L^{(j)}$ then either $s = t$ or $s \notin L^{(j)}$.

2. If $g^0 \neq 0$, then

$$g^0 = \lambda^0 \left( \prod_{s \in \eta} h^0_s(s) \right) \left( \prod_{s \in \lambda} h^0_s(s) \right)$$

where $h^0_s(s)$ is a choice of $d^0_\eta(s)$, $d^0(s)$, $-1$ or $1$, $h^0_s(s)$ is a choice of $d^0_\lambda(s)$, $d^0(s)$, $1$ or $-1$, and

$$\chi^0 = \alpha^m \sum_{T} d^0_T \prod_{i=1}^{m} \tilde{\chi}^{(T(s_m))}(\alpha^{(s_m)}),$$

where the sum is over all 0-admissible tableaux of shape $\nu$ and $s_1, \ldots, s_m$ are the boxes in the diagram of $\nu$.

It is interesting to notice that $\chi^0$ will be a polynomial of degree at most $2|\nu|$ that may or may not factor, as we have seen already for $|\nu| = 1$. Although (3.81) is not a combinatorial formula in the sense that there can be cancellations among the terms of (3.82), its understanding may shed some light in the full Littlewood-Richardson rule. We hope that at least it will help us understand the full Pieri rule.
References


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