XVIII Encontro Brasileiro de Topologia

Minicurso (mini-course)

POINCARÉ DUALITY IN LOW DIMENSIONS

JONATHAN A. HILLMAN

de 29 de julho a 03 de agosto de 2012

Panorama Hotel & Spa

Águas de Lindóia/SP

1991 *Mathematics Subject Classification.* 57P10, 57Mxx,57N13,20Jxx.

Key words and phrases. aspherical, Euler characteristic, fundamental group, intersection pairing, manifold, Poincaré duality.

The goal of this course is to show how Poincaré duality imposes contraints on and interactions between the two most basic invariants of algebraic topology, namely, the Euler characteristic $\chi(M)$ and the fundamental group $\pi_1(M)$, for manifolds M of dimension 3 or 4. (In lower dimensions these invariants determine each other, while in higher dimensions they are decoupled.)

The following notes include material that I may not present in the lectures. In particular, I shall consider only orientable manifolds and PD-complexes, although most of the results below extend to the non-orientable case, and shall not go through proofs of technical points. I shall also try to avoid spectral sequence arguments, but the alternatives are often tedious. (See Lemma 9 below – I may use the spectral sequence approach in the lecture.)

1. presentations and 2-complexes

Every finitely presentable group is the fundamental group of a finite 2-complex. More precisely, a finite presentation $\mathcal{P} = \langle X|R\rangle$ for a group G gives a recipe for constructing a finite complex

$$C(\mathcal{P}) = \vee_{x\in X} S^1_x \cup_{r\in R} e^2_r$$

with one 0-cell, one 1-cell for each generator $x \in X$ and one 2-cell for each relator $r \in R$, and such that $\pi_1(C(\mathcal{P})) = G$.

Example. If $\mathcal{P} = \langle x, y|xyx^{-1}y^{-1}\rangle$ then $G = \mathbb{Z}^2$ and $C(\mathcal{P})$ is the torus.

We may calculate the *Euler characteristic* by counting cells:

$$\chi(C(\mathcal{P})) = 1 - g + r = 1 - \mathrm{def}(\mathcal{P}),$$

where $\mathrm{def}(\mathcal{P}) = g - r$ is the *deficiency* of the presentation. Since $\chi(C(\mathcal{P})) = 1 - \beta_1(C(\mathcal{P})) + \beta_2(C(\mathcal{P}))$ and $H_1(C(\mathcal{P});\mathbb{Z})) = G^{ab}$ (the abelianization of G), by the Hurewicz Theorem, we see that $\chi(C(\mathcal{P}))$ is bounded below by $1-\mathrm{rank}(G^{ab})$. However, we may increase χ without bound, by adding redundant relators.

Conversely, every finite 2-complex with one 0-cell arises from such a presentation. More generally, given any connected 2-complex X, collapsing a maximal tree T in the 1-skeleton of X gives a homotopy equivalent 2-complex X/T with one 0-cell. Thus the study of group presentations is equivalent to the study of connected finite 2-complexes.

Such 2-complexes usually have singularities along the 1-skeleton.

Example. If $\mathcal{P} = \langle x|x^3\rangle$ then points in the 1-skeleton have neighbourhoods which look like $\mathsf{Y} \times (-1, 1)$.

When is G realizable by a manifold? We may thicken up the above construction by using handles instead of cells, to build a handlebody.

We start with a 0-handle (n-disc) $h^0 = D^0 \times D^n$, corresponding to an 0-cell. If $n \geq 2$ the boundary S^{n-1} is infinite, and we may adjoin g 1-handles $h^1 = D^1 \times D^{n-1}$ along disjoint copies of $S^0 \times D^{n-1}$ to obtain

$$\natural^g S^1 \times D^{n-1} = h^0 \cup gh^1 = D^0 \times D^n \cup gD^1 \times D^{n-1}.$$

If $n \geq 4$ we may then represent the relators by disjoint simple closed curves in $\partial(\natural^g S^1 \times D^{n-1}) = \#^g S^1 \times S^{n-2}$. These have product neighbourhoods, and so we may attach 2-handles $h^2 = D^2 \times D^{n-2}$ along disjoint copies of $S^1 \times D^{n-2}$ to get a compact bounded n-manifold

$$N(\mathcal{P}) = h^0 \cup gh^1 \cup rh^2 = D^0 \times D^n \cup gD^1 \times D^{n-1} \cup rD^2 \times D^{n-2}$$

which is homotopy equivalent to $C(\mathcal{P})$. Doubling $N(\mathcal{P})$ along its boundary gives a closed n-manifold $M(\mathcal{P}) = N(\mathcal{P}) \cup_\partial N(\mathcal{P})$, with $\pi_1(M(\mathcal{P})) \cong G$.

If n is odd then $\chi(M) = 0$, and if n is even and ≥ 6 we can modify the construction to vary χ up or down. Thus in high dimensions the fundamental group and the Euler characteristic are independent, and all finitely presentable groups occur.

Our interest is in the low dimensional cases. The cases $n = 1$ and $n = 2$ are well-known. We shall see that Poincaré duality strongly limits the possible groups when $n = 3$, and imposes useful relations between χ and π in dimension 4.

2. Homology and cohomology with local coefficients

In order to define Poincaré duality properly, we shall need to consider all covering spaces. We do this by using cohomology with coefficients in the group ring of the fundamental group.

Let X be a connected finite cell complex with fundamental group G. We may lift the cellular decomposition of X to an equivariant cellular decomposition of the universal covering space $\widetilde{X}$. The cellular chain complex of $\widetilde{X}$ is then a complex $C_* = C_*(\widetilde{X})$ of left $\mathbb{Z}[G]$-modules, with respect to the action of the covering group G. A choice of lifts of the q-cells of X determines a basis for C_q, for all q, and so C_* is a complex of finitely generated free $\mathbb{Z}[G]$-modules.

The i^{th} *equivariant homology* module of X with coefficients $\mathbb{Z}[G]$ is the left module $H_i(X;\mathbb{Z}[G]]) = H_i(C_*)$, which is clearly isomorphic to $H_i(\widetilde{X})$ as an abelian group, with the action of the covering group determining its $\mathbb{Z}[G]$-module structure. The i^{th} *equivariant cohomology* module of X with coefficients $\mathbb{Z}[G]$ is the right module $H^i(X;\mathbb{Z}[G]) = H^i(C^*)$, where $C^* = Hom_{\mathbb{Z}[G]}(C_*, \mathbb{Z}[G])$ is the associated cochain complex of *right* $\mathbb{Z}[G]$-modules. We may switch between right and left modules by using an anti-involution of $\mathbb{Z}[G]$. (Usually the involution is induced by $g \mapsto g^{-1}$.) Let $\overline{N}$ denote the left module associated to a right module N.

More generally, if A and B are right and left $\mathbb{Z}[G]$-modules (respectively) we may define $H_j(X;A) = H_j(A \otimes_{\mathbb{Z}[G]} C_*)$ and $H^{n-j}(X;B) = H^{n-j}(Hom_{\mathbb{Z}[G/H]}(C_*, B))$. If F is a field and the action is trivial the cohomology is the linear dual of the homology $H^j(X;F) \cong H_j(X;F)^* = Hom_F(H_j(X;F), F)$.

A space X is *aspherical* if $\widetilde{X}$ is contractible. We write $X \simeq K(G,1)$, since the homotopy type of X is then determined by its fundamental group. Given any cell complex M, we may construct such an *Eilenberg-Mac Lane* $K(G,1)$-complex by adjoining cells of dimensions ≥ 3 to kill off the higher homotopy groups. The inclusion $c_M : M \to K(G,1)$ is the *classifying map* for the fundamental group. It induces an isomorphism on fundamental groups.

Theorem (Hopf). *The classifying map c_M induces an epimorphism $H_2(c_M) : H_2(M) \to H_2(G) = H_2(K(G,1))$ and a monomorphism $H^2(c_M) : H^2(G) \to H^2(M)$.* □

If $H_2(c_M)$ is an isomorphism then so is $H^2(c_M)$.

[The groups $H_i(G;A)$ and $H^j(G;B)$ may be defined algebraically, in terms of derived functors of tensor product and dual, respectively. Thus $H_i(G;A) = Tor_i^{\mathbb{Z}[G]} A \otimes_{\mathbb{Z}[G]} \mathbb{Z}$ and $H^j(G;B) = Ext^j_{\mathbb{Z}[G]}(\mathbb{Z}, B)$. The analogue for local coefficients of the duality between homology and cohomology is the *Universal Coefficient Spectral Sequence*:

$$E_2^{pq} = Ext^q_{\mathbb{Z}[G]}(H_p(X;\mathbb{Z}[G]), B) \Rightarrow H^{p+q}(X;B),$$

with r^{th} differential d_r of bidegree $(1-r, r)$. In particular, there is an "evaluation" homomorphism from

$$H^j(X;\mathbb{Z}[G]) \to Hom_{\mathbb{Z}[\pi]}(H_j(X;\mathbb{Z}[G]), \mathbb{Z}[G]),$$

but this may be neither one-to-one nor onto. See [5].]

The group G *has cohomological dimension* n if $H^j(G;A) = 0$, for all $j > n$ and all left $\mathbb{Z}[G]$-modules A, while $H^n(G;A) \neq 0$ for some such module. We write $c.d.G = n$. If there is a finite $K(G,1)$-complex then $c.d.G < \infty$, and G is torsion-free.

Groups of cohomological dimension 1 are free. I shall mention here two difficult open problems about groups of cohomological dimension 2. The Eilenberg-Ganea Conjecture is that if $c.d.G = 2$ then there is a 2-dimensional $K(G,1)$-complex. (The corresponding result is true in all other dimensions.) The Whitehead Conjecture is that every connected subcomplex of an aspherical 2-complex is aspherical. Bestvina and Brady have shown that these conjectures cannot both be true!

The book [5] is the standard reference for this topic.

3. ends

If G is finite then $H^i(G;\mathbb{Z}[G])\cong\mathbb{Z}$, for $i=0$, and is 0 otherwise. If G is infinite $H^0(G;\mathbb{Z}[G])=0$. We shall call the next cohomology group $H^1(G;\mathbb{Z}[G])$ the *end module for* G, since it measures the number of ends $e(G)$ of the universal cover of any finite complex X with fundamental group G. [The set of ends of X measures the "number of components of X at infinity": it is the inverse limit of the system of path-components of complements of compact subsets $\{\pi_0(X\setminus K)\}$, where $K\subseteq X$ is compact. For instance, if X is compact it has 0 ends, $\mathbb{R}$ has 2 ends, $\mathbb{R}^2$ has 1 end, and the universal cover of the figure eight $S^1\vee S^1$ has infinitely many ends. If X is the universal cover of a finite cell complex (or compact manifold) these are the only possibilities.]

In algebraic terms, if G is a finitely generated infinite group then $H^1(G;\mathbb{Z}[G])$ is 0, $\mathbb{Z}$ or $\mathbb{Z}^\infty$, and $e(G)=1$, 2 or ∞. If $H<G$ is a subgroup of finite index then the covering space X_H corresponding to H is again a finite complex, and $\widetilde{X_H}=\widetilde{X}$, and so $e(H)=e(G)$.

The group G has more than one end if and only if $G\cong\pi\mathcal{G}$ is the fundamental group of a finite graph $\mathcal{G}$ of finitely generated groups in which at least one edge group is finite. In particular, $e(G)=2$ if and only if G has an infinite cyclic normal subgroup of finite index. If G is torsion-free then $e(G)=\infty$ if and only if G is decomposable as a nontrivial free product. (See [9] for "graphs of groups".)

Partial results of a similar but weaker nature are known for the higher cohomology groups $H^i(G;\mathbb{Z}[G])$ with $i\geq 2$. In particular:

Theorem (Farrell)**.** *Let G be a finitely presentable group with an element of infinite order. Then $H^2(G;\mathbb{Z}[G])$ is 0, $\mathbb{Z}$ or is not finitely generated.* □

(It is expected that $H^2(G;\mathbb{Z}[G])=0$, $\mathbb{Z}$ or $\mathbb{Z}^\infty$.)

Theorem (Bowditch [4])**.** *Let G be a finitely presentable group such that $H^2(G;\mathbb{Z}[G])\cong\mathbb{Z}$. Then G has a subgroup of finite index which is the fundamental group of an aspherical closed surface.* □

See Chapter 13 of [10] for a proof of Farrell's Theorem and connections between ends and cohomology.

4. poincaré duality

Poincaré duality complexes represent the homotopy types of manifolds, and were introduced by Wall. When studying the algebraic topology of manifolds it is natural to work with this broader class of spaces.

If X is a manifold, Poincaré duality gives isomorphisms between homology and cohomology in complementary dimensions. Geometrically, this corresponds to the intersection of submanifolds of complentary dimensions, but we shall use a more algebraic approach. Smooth manifolds can be triangulated. More generally, every closed TOP manifold has the homotopy type of a finite complex. If K is a connected finite n-dimensional complex, $w : \pi = \pi_1(K) \to \{\pm 1\}$ is a homomorphism and $C_* = C_*(\widetilde{K})$, let DC_* be the dual chain complex with $DC_q = \overline{Hom_{\mathbb{Z}[\pi]}(C_{n-q}, \mathbb{Z}[\pi])}$ given by dualizing, defining a left module structure by $(g\delta)(c) = w(g)\delta(c)g^{-1}$ for all $g \in \pi$, $\delta \in DC_q$ and $c \in C_{n-q}$, and re-indexing. Then

K is a PD_n-complex with orientation character $w_1(K) = w$

if $H_n(\mathbb{Z}^w \otimes_{\mathbb{Z}[\pi]} C_*) \cong \mathbb{Z}$ and slant product with an n-cycle which generates this group gives a chain homotopy equivalence $DC_* \simeq C_*$. We shall usually assume that K is orientable ($w = 1$), for simplicity.

A finitely presentable group G is a *PD_n-group* (of type FF) if there is an aspherical finite PD_n-complex with group G. Bieri and Eckmann have given an alternative characterization of such groups.

Theorem. *A finitely presentable group is a PD_n-group (of type FF) $\Leftrightarrow$ c.d.$G = n$, $H^i(G;\mathbb{Z}[G]) = 0$ for $i < n$ and $H^n(G;\mathbb{Z}[G]) \cong \mathbb{Z}$, and the augmentation module $\mathbb{Z}$ has a finite resolution by finitely generated free left $\mathbb{Z}[G]$-modules.*

The necessity of these conditions follows immediately from Poincaré duality for $K(G, 1)$.

In the algebraic study of PD_n-groups it is usual to relax the finiteness conditions on $K(G, 1)$, by dropping the hypotheses that G be finitely presentable and allowing $\mathbb{Z}$ to have a finite projective resolution. With this broader definition, all PD_n-groups are finitely generated, but in every dimension $n \geq 4$ there are PD_n-groups which are not finitely presentable [8]. No such group can be the fundamental group of a closed manifold.

In the lowest dimensions ($n \leq 2$) all PD_n-complexes are standard, i.e., are homotopy equivalent to S^1 or to a closed surface. This was first proven by Eckmann, Müller and Linnell, but also follows from the above theorem of Bowditch. Higher dimensional considerations suggest another, more topological strategy, which can be justified *a posteriori*. The bordism Hurewicz homomorphism from $\Omega_n(X)$ to $H_n(X;\mathbb{Z})$ is an epimorphism in degrees $n \leq 4$. Therefore if X is an orientable PD_n-complex with $n \leq 4$ there is a degree-1 map $f : M \to X$ with domain a closed orientable n-manifold. If X is a finite PD_2-complex then

f *is a homotopy equivalence* $\Leftrightarrow$ $\mathrm{Ker}(\pi_1(f)) = 1 \;\Leftrightarrow\; \chi(M) = \chi(X)$.

If $\mathrm{Ker}(\pi_1(f))$ contains the class of a non-separating simple closed curve γ we may reduce $\chi(M)$ by surgery on γ. Gabai has shown that if X is also a surface then there is such a curve γ. Can this be shown directly?

If $n = 3$ it suffices to know that there there is *some* chain homotopy equivalence $DC_* \simeq C_*$.

Theorem 1. *Let K be a connected finite 3-complex. If $C_* = C_*(\widetilde{K})$ is chain homotopy equivalent to DC_* then K is a PD_3-complex.*

Proof. Let $C_* \otimes_{\mathbb{Z}} C_*$ have the diagonal left π-action, and let $\tau(x \otimes y) = (-1)^{pq} y \otimes x$ for all $x \in C_p$ and $y \in C_q$. Let $\Delta : C_* \to C_* \otimes_{\mathbb{Z}} C_*$ be an equivariant diagonal. Then $\tau\Delta$ is also a diagonal homomorphism, and so is chain homotopic to Δ. Let $\kappa \in C_3$ be a 3–chain such that $1 \otimes \kappa$ is a cycle representing a generator $[K]$ of

$$H_3(\mathbb{Z} \otimes_{\mathbb{Z}[\pi]} C_*) \cong H_3(\mathbb{Z} \otimes_{\mathbb{Z}[\pi]} DC_*) = H^0(C^*; \mathbb{Z}) \cong \mathbb{Z},$$

and let $\Delta(\kappa) = \Sigma x_i \otimes y_{3-i}$. Slant product with $1 \otimes \kappa$ defines a chain map $\theta_* : DC_* \to C_*$ by $\theta(\phi) = \Sigma \phi(x_{3-j}) y_j$ for all $\phi \in DC_j$. The double dual DDC_* is naturally isomorphic to C_*, and the "symmetry" of Δ with respect to the transposition τ implies that $D\theta_*$ and θ_* are chain homotopic.

Suppose first that π is finite. Then $H^0(C^*) \cong \mathbb{Z}$ and $H^1(C^*) = 0$, so $H_2(C_*) = H_1(C_*) = 0$ and $H_3(C_*) \cong \mathbb{Z}$. Therefore $\widetilde{K} \simeq S^3$ and so K is a PD_3–complex by [19].

If π is infinite $H_3(DC_*) = H^0(C^*) = 0$. Since $H_1(DC^*) = H_1(C_*) = 0$ and $H_0(DC_*) = H^3(C^*) \cong H_0(C_*) \cong \mathbb{Z}$, $H_i(\theta_*)$ is an isomorphism for all $i \neq 2$. In particular, since $H_0(\theta_*)$ is an isomorphism the dual $\theta^* : C^* \to DC^*$ also induces an isomorphism

$$H^1(C^*) \cong Ext^1_{\mathbb{Z}[\pi]}(H_0(C_*), \mathbb{Z}[\pi]) \cong Ext^1_{\mathbb{Z}[\pi]}(H_0(DC_*), \mathbb{Z}[\pi]) \cong H^1(DC^*).$$

Hence $H_2(\theta_*) = H_2(D\theta_*)$ is also an isomorphism, and so θ is a chain homotopy equivalence. Therefore K is a PD_3-space. ∎

A similar (and easier) result is true for complexes of dimension 1 or 2. On the other hand, the 1-connected space $S^2 \vee S^4$ is not a PD_4–complex, although it has a cell structure with just three cells, and its cellular chain complex is obviously isomorphic to its linear dual.

5. ORIENTABLE PD_3-COMPLEXES

The goal of 3-manifold theory for the last 50 years or so has been to show that the fundamental group is a complete invariant; the topology of a 3-manifold is determined by its fundamental group. (This is not quite correct if there are lens space summands.)

If M is a closed orientable 3-manifold and $\pi_2(M) \neq 0$ then M contains an S^2 which does not bound a ball, by the *Sphere Theorem* of J.H.C.Whitehead. Hence either $M = S^1 \times S^2$ or M is a proper connected sum. The *Kneser-Milnor Theorem* asserts that every closed orientable 3-manifold has an essentially unique factorization into indecomposable summands. These are either aspherical, $S^1 \times S^2$ or have finite fundamental group. Moreover,

M is indecomposable as a connected sum $\Leftrightarrow \pi$ is indecomposable as a free product $\Leftrightarrow \pi$ has finitely many ends.

Wall asked to what extent these results hold for PD_3-complexes. The answer is "almost, but not quite".

Finite coverings of PD_n-complexes are again PD_n-complexes. Thus when $n = 3$ and $\pi = \pi_1(X)$ is finite $\widetilde{X} \simeq S^3$. If π is infinite then $\pi_2(X) = H_2(\widetilde{X};\mathbb{Z}) \cong H^1(\pi;\mathbb{Z}[\pi])$, while $H_3(\widetilde{X};\mathbb{Z}) = H^0(\pi;\mathbb{Z}[\pi]) = 0$, by the Hurewicz Theorem and Poincaré duality. Thus either π has more than one end or $\widetilde{X}$ is contractible, and so X is aspherical.

The *fundamental triple* of a PD_3-complex P is $(\pi_1(P), w_1(P), c_{P*}[P])$. This is a complete homotopy invariant for such complexes.

Theorem (Hendriks)**.** *Two PD_3-complexes are homotopy equivalent if and only if their fundamental triples are isomorphic.* □

Turaev characterized the possible triples corresponding to a given finitely presentable group and orientation character. In particular, he characterized the pairs (π, w), in terms of associated modules. We shall give a proof for the orientable case $w = 1$.

If G is a finitely presentable group then the augmentation ideal $I_G = \mathrm{Ker}(\varepsilon : \mathbb{Z}[G] \to \mathbb{Z})$ has a finite presentation, with $g \times r$ presentation matrix P, say. The module J_G with presentation matrix $\overline{P}^{tr}$ is well-defined up to direct sum with finitely generated free modules. Let $[J_G]$ be its equivalence class with respect to such stabilization.

Theorem 2. *Let π be a finitely presentable group. Then there is a finite orientable PD_3-complex K with $\pi_1(K) \cong \pi$ if and only if $[I_\pi] = [J_\pi]$.*

Proof. Let K be a connected PD_3-complex with $\pi_1(K) \cong \pi$. We may assume that K has a single 0-cell and finite 2-skeleton, and that C_* and DC_* are finitely generated free $\mathbb{Z}[\pi]$-complexes. Then $C_0 \cong \mathbb{Z}[\pi]$

and $\mathrm{Cok}(\partial_2^C) = \mathrm{Im}(\partial_1^C)$ is the augmentation ideal I_π. The Fox-Lyndon free differential calculus gives a matrix M for ∂_2^C with respect to the bases represented by chosen lifts of the cells of K. Since $H_0(C_*) \cong H_0(DC_*) \cong \mathbb{Z}$ and $I_\pi = \mathrm{Cok}(\partial_2^C)$, Schanuel's Lemma implies that $I_\pi \oplus DC_0 \cong \mathrm{Cok}(\partial_2^D) \oplus C_0$. Since ∂_2^D has matrix $\overline{M}^{tr}$ it follows that $[I_\pi] = [J_\pi]$.

Conversely, let K be the finite 2-complex associated to a presentation for π, and define J_π by means of the Fox-Lyndon matrix. Suppose that $J_\pi \oplus \mathbb{Z}[\pi]^m \cong I_\pi \oplus \mathbb{Z}[\pi]^n$. Let $L = K \vee mD^3$ be the 3-complex obtained by subdividing the 1-skeleton of K at n points distinct from the basepoint and giving each of the 3-discs the cell structure $D^3 = e^0 \cup e^2 \cup e^3$. Then $L \simeq K$ and $\mathrm{Cok}(\partial_2^L) \cong I_\pi \oplus \mathbb{Z}[\pi]^n$. Let $DC_1 = \overline{Hom_{\mathbb{Z}[\pi]}(C_2(\widetilde{L}), \mathbb{Z}[\pi])}$ and let $\alpha : DC_1 \to \mathbb{Z}[\pi]$ be the composite of the projection onto $J_\pi \oplus \mathbb{Z}[\pi]^m$, the isomorphism with $I_\pi \oplus \mathbb{Z}[\pi]^n$, the projection onto I_π and the inclusion into $\mathbb{Z}[\pi]$. Then $\bar{\alpha}^{tr} : \mathbb{Z}[\pi] \to C_2(\widetilde{L})$ has image in $\pi_2(L) = H_2(C_*(\widetilde{L}))$ and so we may attach another 3-cell along a map f in the homotopy class of $\bar{\alpha}^{tr}(1)$. The resulting 3-complex $X = L \cup_f e^3$ satisfies the hypothesis of Theorem 1, and so X is a finite PD_3-complex with fundamental group π. □

Turaev used his result to deduce a basic splitting theorem.

Theorem (Turaev). *A PD_3-complex is irreducible with respect to connected sum if and only if its fundamental group is indecomposable with respect to free product.* □

It follows that orientable PD_3-complexes have essentially unique factorizations.

Crisp showed that the indecomposables are almost as expected from the knowledge of 3-manifolds. We say that a group G is *virtually free* if it has a subgroup of finite index which is a free group.

Theorem (Crisp). *Let P be an indecomposable PD_3-complex. Then either P is aspherical or $\pi_1(P)$ is virtually free.* □

However it is no longer true that the groups of indecomposable complexes need have finitely many ends. There are indecomposable PD_3-complexes with fundamental group $S_3 *_{Z/2Z} S_3$. This group is indecomposable, but has a free subgroup of index 6. No such complex is homotopy equivalent to a 3-manifold.

Crisp's argument starts from the observation that if P is a PD_3-complex then $\pi_2(P) = H_2(P; \mathbb{Z}[\pi]) \cong \overline{H^1(\pi; \mathbb{Z}[\pi])}$, the end-module of the fundamental group. Thus we may study the action of π on $\pi_2(P)$ in two ways. Firstly, since π is finitely presentable it is the fundamental group $\pi \cong \pi\mathcal{G}$ of a finite graph $\mathcal{G}$ of finitely generated groups in which each vertex group has at most one end and each edge group is finite. In other words, it may be assembled from such vertex groups by iterating the operations of free product with amalgamation and HNN extensions over finite subgroups. This decomposition of π leads to a "Mayer-Vietoris" presentation for $H^1(\pi;\mathbb{Z}[\pi])$.

Secondly, we have a simple lemma.

Lemma 3. *Let P be a finite dimensional complex with fundamental group π and such that $H_q(\widetilde{P};\mathbb{Z}) = 0$ for all $q > 2$. If G is a subgroup of π then $H_{s+3}(G;\mathbb{Z}) \cong H_s(G;\pi_2(P))$ for all $s \geq 1$.*

Proof. Asumme that P has dimension n, and let C_* be the cellular chain complex of the universal cover $\widetilde{P}$, considered as a chain complex over $\mathbb{Z}[G]$. Let $Z_q \leq C_q$ be the submodule of q-cycles. Then there are exact sequences

$$0 \to Z_2 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

$$\text{and} \quad 0 \to C_n \to \cdots \to C_3 \to Z_2 \to \pi_2(P) \to 0,$$

since $H_0(C_*) = \mathbb{Z}$, $H_1(C_*) = 0$, $H_2(C_*) = H_2(\widetilde{P};\mathbb{Z}) \cong \pi_2(P)$ and $H_q(\widetilde{P};\mathbb{Z}) = 0$ for all $q > 2$. Elementary homological algebra ("devissage") applied to the first sequence gives $H_{s+3}(G;\mathbb{Z}) \cong H_s(G;Z_2)$ for all $s \geq 1$, and applied to the second sequence gives $H_s(G;Z_2) \cong H_s(G;\pi_2(P))$ for all $s \geq 1$ also. □

Crisp plays off these two facets of the group action to obtain his result. One useful corollary is the following theorem.

Theorem (Crisp)**.** *If X is a PD_3-complex and $g \in \pi = \pi_1(X)$ has prime order p and infinite centralizer $C_\pi(g)$ then $p = 2$, g is orientation-reversing and $C_\pi(g)$ has two ends.* □

Using this result, and some elementary group theory, it is possible to show that if X is an indecomposable PD_3-complex and $\pi = \pi_1(X)$ is the fundamental group of a reduced finite graph of finite groups but is neither $\mathbb{Z}$ nor $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ then X is orientable, the underlying graph is a tree, the vertex groups have cohomological period dividing 4 and all but at most one of the edge groups is $\mathbb{Z}/2\mathbb{Z}$. If there are no exceptions then all but at most one of the vertex groups is dihedral of order $2m$ with m odd. Every such group is realized by some PD_3-complex. Otherwise, one edge group may be $\mathbb{Z}/6\mathbb{Z}$. We do not know of any such examples.

The fact that the topology of a 3-manifold is (essentially) determined by its fundamental group may be regarded as a uniqueness result. However the corresponding existence result "which groups are 3-manifold groups?" has been much less studied. The fundamental groups of aspherical 3-manifolds are PD_3-groups. Thus the major open question is whether every such group is a 3-manifold group, equivalently, whether every aspherical PD_3-complex is homotopy equivalent to a 3-manifold.

If X is a PD_3-complex there is a degree-1 map $f : M \to X$ with domain a closed orientable 3-manifold, by the bordism result mentioned in §4. One would hope to modify the map by "surgery" to obtain a homotopy equivalence. This is not possible in all cases, as the above examples show, but may be true if we also allow passage to finite covers.

6. Homotopy equivalences of 4-manifolds

In the first lecture we saw that every finitely presentable group π is the fundamental group of some closed orientable 4-manifold M. This has the somewhat negative consequence that there can be no algorithmic classification of all closed 4-manifolds. However if we assume the group fixed we may side-step this issue, and in many cases may hope to characterize the homotopy type in terms of familiar invariants. (This seems much less likely in higher dimensions.)

In this section we shall give a criterion for a map between 4-manifolds to be a homotopy equivalence, and derive a criterion for a 4-manifold to be aspherical. We shall need some algebra. If R is a ring and P is a finitely generated R-module then P is projective, with projective complement Q, if $P \oplus Q$ is a free module. The projective module P is *stably free* if $P \oplus R^a \cong R^b$, for some $a, b \geq 0$. If $R = \mathbb{Z}[\pi]$ is a group ring then such stably free R-modules have well defined rank $r(P) = b - a$, $r(P) \geq 0$, and $r(P) = 0 \Leftrightarrow P = 0$. (This result, "group rings are weakly finite", is due to Kaplansky and relies on analysis for its proof!)

If $f : M \to N$ is a homotopy equivalence of closed orientable 4-manifolds then $\pi_1(f)$ is an isomorphism, $H_4(f)$ is an isomorphism and $\chi(M) = \chi(N)$. These simple necessary conditions are also sufficient. For if $\pi_1(f)$ and $H_4(f)$ are isomorphisms then $H_2(f)$ is onto, so $\chi(N) \geq \chi(M)$. Moreover, $\pi_2(f)$ is also onto, and the kernel is a stably free $\mathbb{Z}[\pi]$-module, of stable rank $\chi(N)-\chi(M)$. Thus f is a homotopy equivalence if also $\chi(N) = \chi(M)$.

In more detail:

Lemma 4. *Let R be a ring and C_* be a finite chain complex of projective R-modules. If $H_i(C_*) = 0$ for $i < q$ and $H^{q+1}(Hom_R(C_*, B)) = 0$ for any left R-module B then $H_q(C_*)$ is projective. If moreover $H_i(C_*) = 0$ for $i > q$ then $H_q(C_*) \oplus \bigoplus_{i\equiv q+1\ (2)} C_i \cong \bigoplus_{i\equiv q\ (2)} C_i$.*

Proof. We may assume without loss of generality that $q = 0$ and $C_i = 0$ for $i < 0$. We may factor $\partial_1 : C_1 \to C_0$ through $B = \mathrm{Im}\partial_1$ as $\partial_1 = j\beta$, where β is an epimorphism and j is the natural inclusion of the submodule B. Since $j\beta\partial_2 = \partial_1\partial_2 = 0$ and j is injective $\beta\partial_2 = 0$. Hence β is a 1-cocycle of the complex $Hom_R(C_*, B)$. Since $H^1(Hom_R(C_*, B)) = 0$ there is a homomorphism $\sigma : C_0 \to B$ such that $\beta = \sigma\partial_1 = \sigma j\beta$. Since β is an epimorphism $\sigma j = id_B$ and so B is a direct summand of C_0. This proves the first assertion.

The second assertion follows by an induction on the length of the complex. □

Theorem 5. *Let M and N be finite orientable PD_4-complexes. If $f : M \to N$ is a map such that $\pi_1(f)$ is an isomorphism and $f_*[M] = \pm[N]$ then $\chi(M) \geq \chi(N)$, and* $\mathrm{Ker}(\pi_2(f))$ *is a stably free module of rank $\chi(M) - \chi(N)$.*

Proof. Up to homotopy type we may assume that f is a cellular inclusion of finite cell complexes, and so M is a subcomplex of N. We may also identify $\pi_1(M)$ with $\pi = \pi_1(N)$. Let $C_*(M)$, $C_*(N)$ and D_* be the cellular chain complexes of $\widetilde{M}$, $\widetilde{N}$ and $(\widetilde{N}, \widetilde{M})$, respectively. Then the sequence

$$0 \to C_*(M) \to C_*(N) \to D_* \to 0$$

is a short exact sequence of finitely generated free $\mathbb{Z}[\pi]$-chain complexes.

By the projection formula $f_*(f^*a \cap [M]) = a \cap f_*[M] = \pm a \cap [N]$ for any cohomology class $a \in H^*(N; \mathbb{Z}[\pi])$. Since M and N satisfy Poincaré duality it follows that f induces split surjections on homology and split injections on cohomology. Hence $H_q(D_*)$ is the "surgery kernel" in degree $q - 1$, and the duality isomorphisms induce isomorphisms from $H^r(Hom_{\mathbb{Z}[\pi]}(D_*, B))$ to $H_{6-r}(\overline{D_*} \otimes B)$, where B is any left $\mathbb{Z}[\pi]$-module. Since f induces isomorphisms on homology and cohomology in degrees ≤ 1, with any coefficients, the hypotheses of Lemma 4 are satisfied for the $\mathbb{Z}[\pi]$-chain complex D_*, with $q = 3$, and so $H_3(D_*) = \mathrm{Ker}(\pi_2(f))$ is projective. Moreover $H_3(D_*) \oplus \bigoplus_{i\ odd} D_i \cong \bigoplus_{i\ even} D_i$. Thus $H_3(D_*)$ is a stably free $\mathbb{Z}[\pi]$-module of rank $\chi(D_*) = \chi(M) - \chi(N)$. □

Corollary 6. *Let M and N be finite orientable PD_4-complexes. A map $f : M \to N$ is a homotopy equivalence if and only if $\pi_1(f)$ is an isomorphism, $f_*[M] = \pm[N]$ and $\chi(M) = \chi(N)$.*

Proof. The conditions are clearly necessary. If they hold then $H_3(D_*)$ has rank 0, so is trivial, and f is a homotopy equivalence. □

This leads to a criterion for asphericity.

Theorem 7. *A closed orientable* 4*-manifold M is aspherical if and only if $\pi = \pi_1(M)$ is a PD_4^+-group of type FF and $\chi(M) = \chi(\pi)$.*

Proof. (Sketch). The conditions are clearly necessary. It is fairly easy to see that they suffice, if also $\beta_2(\pi) > 0$. The classifying map $c_M : M \to K(\pi, 1)$ induces an isomorphism on π_1 and $H_2(c_M)$ is onto, by Hopf's Theorem. If $K(\pi, 1)$ is an orientable PD_4-complex and $\chi(M) = \chi(\pi)$ then $H_2(c_M)$ is an isomorphism. Hence so is $H^2(c_M)$. If $\beta_2(\pi) > 0$ then there are classes $x, y \in H^2(M; \mathbb{Z})$ such that $x \cup y$ generates $H^4(M\mathbb{Z})$. But these classes are in the image of $H^2(\pi; \mathbb{Z})$, and so c_M must have degree 1. □

The full claim requires a more subtle argument, using L^2-cohomology, which I shall not give. L^2-methods also give another criterion which is complete and natural, and is in fact more useful.

Theorem. *Let M be a PD_4-complex with fundamental group π. Then M is aspherical $\Leftrightarrow$ $H^s(\pi; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$ and $\beta_2^{(2)}(M) = \beta_2^{(2)}(\pi)$.* □

7. MINIMAL EULER CHARACTERISTIC

In general, M is far from being uniquely determined by π. If N is a simply connected closed 4-manifold then the connected sum $M\#N$ has the same fundamental group. However $\chi(M\#N) = \chi(M) + \chi(N) - 2$, and so this operation tends to increase the Euler characteristic. (In fact, $\chi(N) > 2$ unless $N \cong S^4$, in which case $M\#N \cong M$.)

Simple estimates show that $\chi(M)$ is bounded below in terms of the Betti numbers of π.

Theorem 8. *Let M be a closed orientable* 4*-manifold with fundamental group π. Then $\chi(M) \geq 2 - 2\beta_1(\pi) + \beta_2(\pi)$.*

Proof. We have $\beta_1(M) = \beta_1(\pi)$ and $\beta_2(M) \geq \beta_2(\pi)$, by the Hurewicz and Hopf theorems, respectively. Since $\beta_3(M) = \beta_1(M)$ and $\beta_4(M) = 1$, by Poincaré duality, the result is clear. □

It is natural to ask what is the minimal value of $\chi(M)$ for a given group, and whether the manifold realizing this minimum is unique (up to homotopy equivalence or homeomorphism).

Example. Suppose $\pi \cong Z^r$. Then $\chi(M) \geq 0$, with equality only if $r = 1, 2$ or 4. Moreover M is aspherical $\Leftrightarrow$ $r = 4$ and $\chi(M) = 0$, and then M is homeomorphic to the 4-torus $\mathbb{R}^4/\mathbb{Z}^4$. (The final assertion uses the work of Freedman on TOP surgery.)

The above theorem does not yet fully answer our question about 4-manifolds of minimal Euler characteristic even for PD_4-groups, for we do not know whether every finitely presentable PD_n-group is the fundamental group of a closed aspherical n-manifold for any dimension $n > 2$. As in the 3-dimensional case, if X is an orientable PD_4-complex there is a degree-1 map $f : M \to X$ with domain a closed orientable 4-manifold. We may then modify f by elementary surgery to make $\pi_1(f)$ an isomorphism. However we can go no further with our present knowledge of (4-dimensional) surgery.

In the final sections we shall outline answers to this question for a large class of groups of interest to low-dimensional topologists.

An old conjecture of Hopf asserts that if M is a closed $2k$-manifold with a metric of non-positive curvature then $(-1)^k\chi(M) \geq 0$. This conjecture is still open, even for 4-manifolds. A more algebraic version is that if M is an aspherical 4-manifold then $\chi(M) \geq 0$.

The Euler characteristic and the deficiency are closely related. It can be shown that if M is aspherical then $\mathrm{def}(\pi) < 1 - \frac{1}{2}\chi(\pi)$. If Hopf's conjecture is true then $\mathrm{def}(\pi_1(M)) \leq 0$. Is $\mathrm{def}(\pi) \leq 0$ for every PD_4-group π? This bound is best possible for groups with $\chi = 0$, since the presentation $\langle a, b \mid ba^2 = a^3b^2,\ b^2a = a^2b^3\rangle$ gives a solvable group $\pi \cong Z^3 \rtimes_A Z$.

8. The intersection pairing

The central geometric manifestation of Poincaré duality is the intersection pairing on the middle-dimensional homology of an even-dimensional manifold.

It has long been known that the homotopy type of a 1-connected 4-manifold M is determined by the intersection pairing on $H_2(M; \mathbb{Z})$, or equyivalently by the cohomology ring $H^*(M; \mathbb{Z})$. Freedman showed that such manifolds are determined up to homeomorphism by their homotopy type and one other invariant, the Kirby-Siebenmann stable smoothing invariant in $Z/2Z$. Moreover, every unimodular symmetric bilinear form on a finitely generated free abelian group is the intersection pairing of some closed TOP 4-manifold. Donaldson showed that in general these may not be smoothable. We shall say no more about this.

When $\pi \neq 1$ it is more useful to consider intersections in the universal cover. There is an equivariant intersection pairing

$$\bullet : \pi_2(M) \times \pi_2(M) \to \mathbb{Z}[\pi]$$

which is additive in each factor and sequilinear with respect to the natural anti-involution of the group ring. This means that

$$(x+x')\bullet y = x\bullet y + x'\bullet y,$$
$$(gx)\bullet y = g(x\bullet y) \quad \text{and}$$
$$y\bullet x = \overline{x\bullet y},$$

for all $g\in\pi$ and $x,x',y\in\pi_2(M)$. There is an associated *evaluation* homomorphism $ev: H^2(M;\mathbb{Z}[\pi])\to\overline{Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi])}$. The intersection pairing is *non-singular* if ev is an isomorphism.

Lemma 9. *Let M be a PD_4-space with fundamental group π and let $\Pi=\pi_2(M)$. Then $\Pi\cong\overline{H^2(M;\mathbb{Z}[\pi])}$ and there is an exact sequence*

$$0\to H^2(\pi;\mathbb{Z}[\pi])\to H^2(M;\mathbb{Z}[\pi]) \xrightarrow{ev} Hom_{\mathbb{Z}[\pi]}(\Pi,\mathbb{Z}[\pi])\to H^3(\pi;\mathbb{Z}[\pi]).$$

Proof. Let C_* be the equivariant chain complex of the universal cover $\widetilde{M}$. Then $H_2(M;\mathbb{Z}[\pi])=H_2(C_*)\cong\Pi$, by the Hurewicz Theorem in degree 2, and so the first assertion follows from Poincaré duality. The exact sequence follows directly from the Universal Coefficient spectral sequence. However we shall give a more elementary argument.

If L is a left $\mathbb{Z}[\pi]$-module, let $e^iL=Ext^i_{\mathbb{Z}[\pi]}(L,\mathbb{Z}[\pi])$, for $i\geq 0$. Then $L^*=e^0L=Hom_{\mathbb{Z}[\pi]}(L,\mathbb{Z}[\pi])$ is the dual right module. The chain complex C_* gives several exact sequences

$$0\to Z_0\to C_0\to\mathbb{Z}\to 0,$$
$$0\to Z_1\to C_1\to Z_0\to 0,$$
$$0\to Z_2\to C_2\to Z_1\to 0,$$
$$\text{and}\quad 0\to Z_3\to C_3\to Z_2\to\Pi\to 0.$$

Applying $Hom_{\mathbb{Z}[\pi]}(-,\mathbb{Z}[\pi])$ to these sequences and using elementary homological algebra, we obtain isomorphisms

$$e^1Z_0\cong e^2\mathbb{Z}=H^2(\pi;\mathbb{Z}[\pi])$$
$$\text{and}\quad e^1Z_1\cong e^2Z_0\cong e^3\mathbb{Z}=H^3(\pi;\mathbb{Z}[\pi])$$

and exact sequences

$$C^1\to Z_1^*\to H^2(\pi;\mathbb{Z}[\pi])\to 0,$$
$$0\to Z_1^*\to C^2\to Z_2^*\to H^3(\pi;\mathbb{Z}[\pi])\to 0$$
$$\text{and}\quad 0\to\Pi^*\to Z_2^*\to C_3^*.$$

If $f\in C^2$ is a cocycle then $f|_{Z_2}$ is in the image of Π^*, so the evaluation homomorphism defined by $ev([f])=[f|_{Z_2}]$ maps H^2 to Π^*. With a little effort we find that the kernel of ev is $H^2(\pi;\mathbb{Z}[\pi])$, while the cokernel is a submodule of $H^3(\pi;\mathbb{Z}[\pi])$. (It helps to label the maps in the original exact sequences!) □

In fact $\mathrm{Cok}(ev) = H^3(\pi;\mathbb{Z}[\pi])$, so the right-hand map is onto, but this is best seen using the spectral sequence approach. (It uses the fact that $H^3(M;\mathbb{Z}[\pi]) \cong H_1(\widetilde{M};\mathbb{Z}) = 0$ by Poincaré duality.)

When π is finite or (more generally) has a free subgroup of finite index then $H^s(\pi;\mathbb{Z}[\pi] = 0$ for all $s > 1$ and so the sequence reduces to an isomorphism $\pi_2(M) \cong \overline{Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi])}$.

We shall say that M is *strongly minimal* if the intersection pairing is trivial, i.e., if $ev = 0$. Strongly minimal PD_4-complexes are minimal with respect to the partial order defined by $M > N$ if there is a map $f : M \to N$ such that $\pi_1(f)$ is an isomorphism and $f_*[M] = \pm[N]$.

9. 4-manifolds with free fundamental group

The easiest classes of 4-manifolds to study after the 1-connected case are those with free fundamental group. If $\pi \cong F(r)$ is free of finite rank r then $\beta_1(\pi) = r$ and $\beta_2(\pi) = 0$. Hence $\chi(M) \geq 2 - 2r$, by Theorem 8. The minimum is realized by $\#^r S^1 \times S^3$, the connected sum of r copies of $S^1 \times S^3$.

Lemma 10. *Let L be a finitely generated (left) $\mathbb{Z}[F(r)]$-module. Then $L^* = Hom_{\mathbb{Z}[F(r)]}(L,\mathbb{Z}[F(r)])$ is a finitely generated free (right) module.*

Proof. This uses three facts about $\mathbb{Z}[F(r)]$-modules. Finitely generated $\mathbb{Z}[F(r)]$-modules are finitely presentable ($\mathbb{Z}[F(r)]$ is *coherent*), projective $\mathbb{Z}[F(r)]$-modules are free, and every $\mathbb{Z}[F(r)]$-module has a free resolution of length at most 2 (since $c.d.F(r) = 1$).

If L is finitely generated then there is an exact sequence

$$\mathbb{Z}[F(r)]^b \to \mathbb{Z}[F(r)]^a \to L \to 0.$$

Dualizing, we get an exact sequence of right modules

$$0 \to L^* \to \mathbb{Z}[F(r)]^a \to \mathbb{Z}[F(r)]^b \to C \to 0,$$

where C is the cokernel of the middle map. Since C is finitely generated it also has a finitely generated free resolution of length at most 2. Therefore L^* is a finitely generated free module, by Schanuel's Lemma. □

Theorem 11. *Let M be a closed orientable 4-manifold with fundamental group $\pi \cong F(r)$. Then $\pi_2(M)$ is a finitely generated free $\mathbb{Z}[F(r)]$-module of rank $\beta = \chi(M) + 2r - 2$, and the equivariant intersection pairing is non-singular.*

Proof. We may assume that M has a cell structure with one 0-cell and c_q q-cells, where $c_q = 0$ for $q > 4$. Let C_* be the equivariant chain complex of the universal cover $\widetilde{M}$, and let B_q and Z_q be the submodules of q-boundaries and q-cycles in C_q. Let $H_q = Z_q/B_q$ and $\Pi = \pi_2(M)$. Then $\Pi \cong H_2$, by the Hurewicz Theorem. There are exact sequences

$$0 \to B_0 \to C_0 \to \mathbb{Z} \to 0,$$

$$0 \to B_1 \to C_1 \to B_0 \to 0,$$

$$0 \to Z_2 \to C_2 \to B_1 \to 0,$$

$$0 \to Z_3 \to C_3 \to Z_2 \to \Pi \to 0,$$

$$\text{and} \quad 0 \to H_4 \to C_4 \to Z_3 \to H_3 \to 0.$$

The map from $C_0 = \mathbb{Z}[\pi]$ to $\mathbb{Z}$ is the augmentation homorphism. Since $\pi = F(r)$ the augmentation ideal is free of rank r, so $B_0 \cong \mathbb{Z}[F(r)]^r$. Therefore the second and third of these sequences are split exact. In particular, B_0, B_1 and Z_2 are finitely generated free $\mathbb{Z}[F(r)]$-modules.

Let C^* be the dual cochain complex, with $C^q = Hom_{\mathbb{Z}[\pi]}(C_q, \mathbb{Z}[\pi])$, and let $H^q = H^q(C^*)$ be the corresponding cohomology module. Then Poincaré duality gives $H_4 = H_4(\widetilde{M}; \mathbb{Z}) = H^0 = 0$ and an isomorphism $H^2 \cong Hom_{\mathbb{Z}[\pi]}(\Pi, \mathbb{Z}[\pi])$. Hence H^2 is a free right module, by Lemma 10, and so $\Pi \cong \overline{H^2}$ is a free left $\mathbb{Z}[F(r)]$-module, by Lemma 9. Thus the fourth sequence also splits. Therefore Z_3 is free and the complex C_* is chain homotopy equivalent to the sum of the three free complexes $B_0 \to C_0$, Π and $C_4 \to Z_3$ (with $C_0 \cong C_4 \cong \mathbb{Z}[F(r)]$, B_0 in degree 1, Π in degree 2 and the degrees otherwise given by the subscripts). Therefore $B_0 \cong Z_3 \cong \mathbb{Z}[F(r)]^r$, $\mathbb{Z} \otimes_\Gamma \Pi \cong H_2(\mathbb{Z} \otimes_\Gamma C_*) \cong H_2(P; \mathbb{Z}) \cong \mathbb{Z}^\beta$, and so $\Pi \cong \mathbb{Z}[F(r)]^\beta$. □

Theorem 12. *If $\beta = 0$ then $M \simeq \#^r S^1 \times S^3$.*

Proof. (Sketch). Let $Q = \#^r S^1 \times S^3$, and let $Q_o = Q \setminus D^4$ be the 3-skeleton of Q. Then $Q_o \simeq (\vee^r S^1) \vee (\vee^r S^3)$. Since $\Pi = 0$ the universal cover of M_o is 2-connected, and $\pi_3(M_o) = H_3(M_o; \mathbb{Z}[F(r)]) \cong \mathbb{Z}[F(r)]^r$. Hence there is a map $j : Q_o \to M_o$ which induces isomorphisms on π_i for $i \leq 3$, and which is therefore a homotopy equivalence. Since every automorphism of $\mathbb{Z}[F(r)]^r$ may be realized by a self homotopy equivalence of Q_o which is the identity on the 1-skeleton $\vee^r S^1$, there is an essentialy unique way to attach the top cell to obtain a PD_4-complex. Hence j extends to a homotopy equivalence $M \simeq Q$. □

We shall state the main result on such manifolds, but not attempt a detailed proof.

Theorem. *Closed orientable 4-manifolds M and M' with fundamental group $F(r)$ are homotopy equivalent $\Leftrightarrow$ their equivariant intersection pairings are isometric.* □

In outline, we first show that the homotopy type of the 3-skeleton $M_o = M\backslash D^4$ is determined by π and χ, and then compare the attaching maps, which differ by an element of $\Gamma_W(\Pi) < \pi_3(M_o)$. This may be related to the intersection pairing through the interpretation of $\Gamma_W(\Pi)$ as the homology in degree 4 of $K(\Pi,2)$, the universal cover of the second Postnikov approximation to M. See [13] for details.

Every such pairing is realized by some PD_4-complex, but it is not yet clear whether it is realized by a closed 4-manifold.

10. 4-manifolds with fundamental group of cohomological dimension 2

The class of groups of cohomological dimension 2 includes surface groups, knot groups, and more generally the groups of compact 3-manifolds with at least one non-spherical boundary component. For these groups the equivariant intersection pairing is again the crucial algebraic invariant after the fundamental group.

Theorem 11 has the following analogue, with a similar proof.

Theorem. *Let M be a closed orientable 4-manifold with fundamental group π, and let $\Pi = \pi_2(M)$. If c.d.$\pi = 2$ then $\chi(M) \geq 2\chi(\pi)$, and $\Pi \cong P \oplus \overline{H^2(\pi;\mathbb{Z}[\pi])}$, where P is a stably free $\mathbb{Z}[\pi])$-module of rank $\chi(M) - 2\chi(\pi)$. The intersection pairing is trivial on $\overline{H^2(\pi;\mathbb{Z}[\pi])}$, and induces a nonsingular pairing on P.* □

There is a 2-connected degree-1 map $f : M \to X$ to a PD_4-complex with minimal Euler characteristic, and $\mathrm{Ker}(\pi_2(f))$ is a projective $\mathbb{Z}[\pi]$-module with a non-singular intersection pairing. Moreover, the "minimal model" X is strongly minimal: its equivariant intersection pairing is trivial.

There is also a result corresponding to the final theorem of §9.

Theorem. *Let M and M' be closed 4-manifolds with 2-connected degree-1 maps $f : M \to X$ and $f' : M' \to X$ to the same strongly minimal PD_4-complex X, with fundamental group π. If c.d.$\pi = 2$ then M and M' are homotopy equivalent $\Leftrightarrow$ the intersection pairings on $\mathrm{Ker}(\pi_2(f))$ and $\mathrm{Ker}(\pi_2(f'))$ are isometric.*

However, we have only been able to determine the minimal models under further hypotheses on the groups. The results suggest that there are finitely many minimal complexes X for each such group, and these are distinguished by their second Wu classes $v_2(X)$ in $H^2(X;\mathbb{F}_2)$.

In a 4-manifold M the class $v_2(M)$ measures self-intersections, and $v_2(M) = 0 \Leftrightarrow$ *every closed surface immersed in M has an even number* *of self-intersections*. It is convenient to distinguish three *Wu types*. A closed 4-manifold M has Wu type

(I) if $v_2(M)$ is not in the image of $H^2(\pi;\mathbb{F}_2)$ in $(H^2(M;\mathbb{F}_2)$;

(II) if $v_2(M)=0$;

(III) if $v_2(M)\neq 0$ but $v_2(M)$ is in the image of $H^2(\pi;\mathbb{F}_2)$.

(No strongly minimal4-manifold has Wu type I.)

If there is a finite 2-dimensional $K(\pi,1)$ complex then there is a closed orientable 4-manifold M with $\chi(M)=2\chi(\pi)$, and $v_2(M)$ may be any class in the image of $H^2(\pi;\mathbb{F}_2)$.

The case of most natural interest is when π is a PD_2^+-group, i.e., $\pi=\pi_1(F)$ where F is a closed orientable surface of genus $g\geq 1$. In this case $\chi(M)\geq 4(1-g)$, and the minimum is realized by two distinct homotopy types: the product $S^2\times F$ (of Wu type II), and the total space of the nontrivial orientable S^2-bundle over F (of Wu type III). In this case we can again give a satisfactory characterization of such manifolds which does not explicitly mention minimal models.

Theorem. *Closed orientable* 4*-manifolds M and M' with fundamental group a PD_2^+-group π are homotopy equivalent $\Leftrightarrow$ their equivariant intersection pairings are isometric, and they have the same Wu type.* □

See [15] for details of the arguments.

11. The references

The references include some papers not cited above, but which are close to the theme of this mini-course. With the exception of [18], which began the modern study of Poincaré duality, I have not included historically important papers (such as those of Bieri and Eckmann on the foundations of the theory of Poincaré duality groups).

The books [5, 9] and [10] are very well-written texts, whereas [2] and [11] are research monographs which assume more background knowledge. The basic references for the case of dimension 3 are [6, 14, 17] and [19]; see also [1, 7, 12, 16] and Chapter 2 of [11]. In dimension 4 the basic references are [13], [15] and Chapter 3 of [11]. (The preprint [15] is a compilation of three earlier papers of mine, which appeared in *Topology and its Applications*.) See also [2, 3].

References

[1] Bleile, B. Poincaré duality pairs of dimension three, Forum Math. 22 (2010), 277–301.

[2] Baues, H.-J. *Combinatorial Homotopy and 4-Dimensional Complexes*, W. De Gruyter, Berlin – New York (1991).

[3] Baues, H.-J. and Bleile, B. Poincaré duality complexes in dimension four, Algebraic and Geometric Topology 8 (2008), 2355-2389.
[4] Bowditch, B.H. Planar groups and the Seifert conjecture, J. Reine Angew. Math. 576 (2004), 11–62.
[5] Brown, K.S. *Cohomology of Groups*, Graduate Texts in Mathematics 87, Springer-Verlag, Berlin - Heidelberg - New York (1982).
[6] Crisp, J.S. The decomposition of Poincaré duality complexes, Commentarii Math. Helvetici 75 (2000), 232–246.
[7] Crisp, J.S. An algebraic loop theorem and the decomposition of PD^3-pairs, Bull. Lond. Math. Soc. 39 (2007), n46–52.
[8] Davis, M.W. The cohomology of a Coxeter group with group ring coefficients, Duke Math. J. 91 (1998), 297–314.
[9] Dicks, W. and Dunwoody, M.J. *Groups acting on Graphs*, Cambridge studies in advanced mathematics 17, Cambridge University Press, Cambridge - New York - Melbourne (1989).
[10] Geoghegan, R. *Topological methods in group theory*, Graduate Texts in Mathematics 243, Springer-Verlag, Berlin - Heidelberg - New York (2008).
[11] Hillman, J.A. *Four-Manifolds, Geometries and Knots*, GT Monographs 5, Geometry and Topology Publications (2002). Latest revision: http://www.maths.usyd.edu.au/u/jonh/ .
[12] Hillman, J.A. Some questions on subgroups of 3-dimensional Poincaré duality groups, http://www.maths.usyd.edu.au/u/jonh/ .
[13] Hillman, J.A. PD_4-complexes with free fundamental group, Hiroshima Math. J. 34 (2004), 295–306.
[14] Hillman, J.A. Indecomposable PD_3-complexes, Alg. Geom. Top. 12 (2012), 131–153.
[15] Hillman, J.A. PD_4-complexes with groups of cohomological dimension 2, http://www.maths.usyd.edu.au/u/jonh/ .
[16] Kapovich, M. and Kleiner, B. Coarse Alexander duality and duality groups, J. Diff. Geom. 69 (2005), 279–352.
[17] Turaev, V.G. Three-dimensional Poincaré complexes: classification and splitting, Math. Sbornik 180 (1989), 809–830.
[18] Wall, C.T.C. Poincaré complexes.I, Ann. Math. 86 (1967), 213–245.
[19] Wall, C.T.C. Poincaré duality in dimension 3, in *Proceedings of the Casson Fest*, GT Monographs 7, Geometry and Topology Publications (2004), 1–26.

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia
E-mail address: jonathan.hillman@sydney.edu.au