WHOLE-LINE SPECTRAL PACKING CONTINUITY THROUGH POWER-LAW SUBORDINACY

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Abstract. By using methods of subordinacy theory, we study packing continuity properties of spectral measures of discrete one-dimensional Schrödinger operators acting on the whole-line. Then, we apply these methods to Sturmian operators with rotation numbers of quasibounded density to show that they have purely α-packing continuous spectrum. A dimensional stability result is also mentioned.

1. Introduction

We are interested in packing-dimensional properties of spectral measures for discrete Schrödinger operators H, in $l^2(\mathbb{Z})$, of the form

$$(H\psi)(n) = \psi(n+1) + \psi(n-1) + V(n)\psi(n), \tag{1.1}$$

with (real) potentials $V = \{V(n)\}$. First, we extend some results from (the partial) packing subordinacy theory for one-dimensional operators on the half-line [3] to the whole-line case. This was initially proposed to provide information about packing dimensional properties of spectral measures and it was an adaptation of the (Hausdorff) power-law subordinacy introduced by Jitomirskaya and Last in [14, 15]. We refer to the latter as Hausdorff subordinacy theory.

The fractal (that is, Hausdorff and packing) subordinacy theories are generalizations of the subordinacy theory, introduced by Gilbert and Pearson in [10, 11] (see [16] for and adaptation to discrete operators). All of them exploit the relation between the asymptotic behavior of the solutions to the eigenvalue equation

$$(H\psi)(n) = E\psi(n) \tag{1.2}$$

and the spectral nature of the operator H. The idea is to use the existence of subordinate or power-law subordinate solutions to (1.2) to investigate the standard decomposition of a spectral measure into its point, singular continuous and absolutely continuous.

Fix $E \in \mathbb{R}$, $\varphi \in (-\pi/2, \pi/2]$, and denote by $u_{1,\varphi,E}$ and $u_{2,\varphi,E}$ the solutions to (1.2) which satisfy the initial conditions

$$\begin{cases} u_{1,\varphi,E}(0) = -\sin\varphi & u_{2,\varphi,E}(0) = \cos\varphi \\ u_{1,\varphi,E}(1) = \cos\varphi & u_{2,\varphi,E}(1) = \sin\varphi \end{cases} . \tag{1.3}$$

A solution ψ to (1.2) is called subordinate at $+\infty$ if

$$\lim_{L\to\infty} \frac{\|\psi\|_L}{\|\Phi\|_L} = 0$$

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holds for any linearly independent solution Φ to (1.2); here, $\|\cdot\|_L$ denotes the norm truncated at $L \in \mathbb{R}$ ($[L]$ is the integral part of L), that is,

$$\|\psi\|_L = \left[\sum_{n=1}^{[L]} |\psi(n)|^2 + (L-[L])|\psi([L]+1)|^2\right]^{\frac{1}{2}} ;$$

the subordinacy of a solution ψ at $-\infty$ is defined analogously.

Given $\alpha \in (0,1]$, a solution ψ to (1.2) is called α-Hausdorff (packing) subordinate at $+\infty$ if

$$\liminf(\limsup)_{L\to\infty} \frac{\|\psi\|_L}{\|\Phi\|_L^{\alpha/(2-\alpha)}} = 0$$

holds for any other linearly independent solution.

In particular, the α-Hausdorff (packing) continuous part of the spectral measure is supported on the set of energies E for which (1.2) does not have α-Hausdorff (resp. packing) subordinate solutions at $-\infty$ or at $+\infty$, and its α-Hausdorff singular part is supported on the set of energies E for which $u_{1,\varphi,E}$ is an α-Hausdorff subordinate solution at both $\pm\infty$ (note the absence of a characterization of the corresponding α-packing singular part; see below).

The possible existence of power-law bounds of the form [5, 7, 15]

$$C_1 L^{\gamma_1} \le \|u\|_L \le C_2 L^{\gamma_2}, \tag{1.4}$$

for positive constants $C_1(E), C_2(E), \gamma_1, \gamma_2$ and every solution u (with normalized initial conditions (NIC), i.e., $|u(0)|^2 + |u(1)|^2 = 1$) to the generalized eigenvalue equation (1.2) imply non-existence of α-Hausdorff subordinate solutions at $+\infty$ (similarly at $-\infty$), with $\alpha = \frac{2\gamma_1}{\gamma_1+\gamma_2}$.

In the following theorem we present a natural version of such tool to prove the lack of α-packing subordinate solutions for some fixed energy E; its proof appears at the end of Section 2.

Theorem 1.1. *Let $\sigma(H)$ be the spectrum of H and let μ_ϕ be the spectral measure of the pair (H,ϕ), with $\phi \in l^2(\mathbb{Z})$. Suppose that there are constants τ_1, τ_2 and a subsequence $L_j \to \infty$ such that, for each $E \in \sigma(H)$, every solution to (1.2) with NIC obeys the estimates*

$$C_1 L_j^{\tau_1} \le \|u\|_{L_j} \le C_2 L_j^{\tau_2}, \tag{1.5}$$

where $C_1 = C_1(E), C_2 = C_2(E)$ are suitable positive constants. Then, H has purely α-packing continuous spectrum, with $\alpha = \frac{2\tau_1}{\tau_1+\tau_2}$, that is, for any $\phi \in l^2$, μ_ϕ is purely α-packing continuous.

Remark 1.2. Similarly to Remark 2 in [7], there is an analogous left half-line version of the previous result. If one is able to establish power-law bounds (1.5) on the restriction of the operator to the right half-line, then the resulting α-packing continuity is independent of the potential on the left half-line. In this sense, the most packing continuous half-line dominates and bounds the dimensionality of the whole-line problem from below.

We apply Theorem 1.1 to the family $\{H_{\lambda,\theta,\rho}\}$ of operators (1.1) with almost periodic Sturmian potentials

$$V(n) = V_{\lambda,\theta,\rho}(n) = \lambda\chi_{[1-\theta,1)}(n\theta + \rho \bmod 1), \quad n \in \mathbb{Z},$$

with coupling constant $0 \neq \lambda \in \mathbb{R}$, irrational rotation number $\theta \in [0,1)$ and (initial) phase $\rho \in [0,1)$.

Recall that any irrational $\theta \in [0,1)$ has an infinite continued fraction expansion

$$\theta = \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots}}} = [0; a_1, a_2, \ldots], \tag{1.6}$$

with uniquely determined $a_n \in \mathbb{N}$. The associated rational approximants p_n/q_n are obtained from

$$p_0 = 0, \quad p_1 = 1, \quad p_n = a_n p_{n-1} + p_{n-2},$$
$$q_0 = 1, \quad q_1 = a_1, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

Definition 1.3. Let $\theta \in [0,1)$ be an irrational number and (1.6) its continued fraction expansion. Then, θ is said to be a number of bounded (quasibounded) density if

$$\limsup(\liminf)_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} a_i < \infty.$$

Remark 1.4. The set of quasibounded density numbers is uncountable, but has Lebesgue measure zero (see page 93 in [17]). We have found that the (rather natural) proposal of the concept of "quasibounded density" is the convenient one in the packing set.

Theorem 1.5. *Let θ be an irrational number of quasibounded density. Then, for every $\lambda \neq 0$, there exists $\alpha = \alpha(\lambda,\theta) > 0$ such that, for every $\phi \in l^2(\mathbb{Z})$, the spectral measure of the pair $(H_{\lambda,\theta,\rho}, \phi)$ is purely α-packing continuous.*

It is well known [5, 7, 15] that each operator $H_{\lambda,\theta,\rho}$, with coupling constant $0 \neq \lambda \in \mathbb{R}$, irrational rotation number of bounded density $\theta \in [0,1)$, and phase $\rho \in [0,1)$, has purely α_H-Hausdorff continuous spectrum (and that $\sigma(H_{\lambda,\theta,\rho})$ has zero Lebesgue measure) for some $\alpha_H \in (0,1)$, with $\alpha_H = \frac{2\gamma_1}{\gamma_1+\gamma_2}$, where $\gamma_1 = \gamma_1(\theta,\lambda) \geq 0, \gamma_2 = \gamma_2(\theta,\lambda) > 0$ satisfy relation (1.4).

Since (1.4) is a particular instance of (1.5), a bounded density rotation number also implies α-packing continuity of the spectral measure of the operator $H_{\lambda,\theta,\rho}$, and here we will get the additional information $\alpha > \alpha_H$ (see Remark 3.7 in Section 3).

We are also interested in the extension, to this packing setting, of the spectral Hausdorff dimensional stability results presented in [1]. In this direction, we have the following

Corollary 1.6. *Let θ be an irrational number of bounded density and γ_1, γ_2 as in (1.4). Then, for every $\rho \in [0,1)$ and $\lambda \neq 0$, the singular continuous component of each spectral measure of the operator*

$$(H^P_{\lambda,\theta,\rho}\psi)(n) := (H_{\lambda,\theta,\rho}\psi)(n) + P(n)\psi(n), \quad \psi \in l^2(\mathbb{Z}), \tag{1.7}$$

with the perturbation P satisfying $|P(n)| \leq C(1+|n|)^{-p}$, for all $n \in \mathbb{Z}$, some $C > 0$ and $p > 3\gamma_2 - \gamma_1$, when it exists, is also purely α-packing continuous.

Remark 1.7. We emphasize that under certain perturbations no singular continuous component may be present, as it is the case of rank one perturbations of operators with singular continuous spectrum of zero Lebesgue measure (this follows directly from results of Simon-Wolff [22]).

Remark 1.8. We also note that the stability of spectral packing dimensional properties for some classes of sparse operators, that is, we obtain packing versions of Theorem 1.2 in [1] (see Section 4 for details).

The organization of this paper is as follows. In Section 2, part of the subordinacy theory is recalled and the proof of Theorem 1.1 is presented. Section 3 is devoted to the proof of Theorem 1.5, after recalling basics of Sturmian potentials. Some packing stability results of operators of the form (1.1), under suitable power decaying perturbations, are discussed in Section 4. For reader convenience, some definitions and concepts regarding Hausdorff and packing measures are recalled in an appendix.

2. Subordinacy theory

Now, we recall some important results of subordinacy theory and use them in order to obtain information about the spectral packing dimensional properties of (1.1). In what follows, we adopt the same strategy presented in [15].

The study of the spectral measure of an operator given by (1.1) is related to the study of the Weyl-Titchmarsh m-functions. To each such whole-line operator H, consider two operators, denoted by $H^{\pm}$, which correspond to the restrictions of (1.1) to $l^2(\mathbb{Z}^{\pm})$, respectively, where $\mathbb{Z}^+ = \{1, 2, \ldots\}$ and $\mathbb{Z}^- = \{0, -1, -2 \ldots\}$. For each $z \in \mathbb{C}\backslash\mathbb{R}$, let $\psi^{\pm}(n; z)$ be the unique solutions to

$$H\psi^{\pm} = z\psi^{\pm}, \quad \text{satisfying} \quad \psi^{\pm}(0; z) = 1 \quad \text{and} \quad \sum_{n=0}^{\infty} |\psi^{\pm}(\pm n; z)|^2 < \infty.$$

With this notation, the m-functions are given, for every $z \in \mathbb{C}\backslash\mathbb{R}$, by

$$m^+(z) = \langle \delta_1 | (H^+ - z)^{-1} \delta_1 \rangle = -\psi^+(1; z)/\psi^+(0; z),$$
$$m^-(z) = \langle \delta_0 | (H^- - z)^{-1} \delta_0 \rangle = \psi^-(1; z)/\psi^-(0; z),$$

where $\delta_j = (\delta_{ij})_{i \geq 1}$. We note that for the whole-line case, the m-function is a matrix-valued function $M(z)$ so that

$$\begin{bmatrix} a & b \end{bmatrix} M(z) \begin{bmatrix} a \\ b \end{bmatrix} = \left\langle (a\delta_0 + b\delta_1) | (H - z)^{-1} (a\delta_0 + b\delta_1) \right\rangle,$$

or, more explicitly (omitting the z dependence)

$$M = \frac{1}{-m^+ - m^-} \begin{bmatrix} 1 & m^+ \\ m^+ & -m^+ m^- \end{bmatrix}.$$

Let $m(z) = \mathrm{tr}(M(z))$, that is, the trace of M. These definitions relate the m-functions to resolvents and hence, to spectral measures. Explicitly, one has

$$m^{\pm}(z) = \int \frac{1}{t - z} d\mu^{\pm}(t),$$
$$m(z) = \int \frac{1}{t - z} d\mu(t),$$

with μ^+ and μ^- representing, respectively, spectral measures of the pairs (H^+, δ_1), (H^-, δ_0), and with $\mu = \mu^+ + \mu^-$, that is, the sum of the spectral measures of the pairs (H^+, δ_1) and (H^-, δ_0). Note that the pair of vectors $\{\delta_0, \delta_1\}$ is cyclic for H.

It was shown in [14] that

$$(\overline{D}^{\alpha} \mu)(E) = \infty \quad \Leftrightarrow \quad \limsup_{\varepsilon \to 0} \varepsilon^{1-\alpha} |m(E + i\varepsilon)| = \infty, \tag{2.1}$$

whereas for the inferior derivative one may only conclude that

$$(\underline{D}^{\alpha}\mu)(E) = \infty \quad \Rightarrow \quad \liminf_{\varepsilon\to 0} \varepsilon^{1-\alpha}|m(E+i\varepsilon)| = \infty. \tag{2.2}$$

There is a mistake in the discussion in [3] (in Theorems 14 there) and currently one guarantees that only the above implication in (2.2) holds true (there is no proof or counterexample to the converse statement). However, we emphasize that (2.2) is exactly what we need in this work.

These results, together with Remark 4.9, show that the study of dimensional spectral properties of Schrödinger operators (1.1) can sometimes be reduced to the study of the behavior of $m(E+i\varepsilon)$ as $\varepsilon \to 0$, which in turn reduces to the study of the behavior of $m^{\pm}(E+i\varepsilon)$ as $\varepsilon \to 0$.

Given an operator H of the form (1.1) and $E \in \mathbb{R}$, let $u^{\pm}_{1,\varphi,E}$ and $u^{\pm}_{2,\varphi,E}$ be the solutions to (1.2), defined in $\mathbb{Z}^{\pm}$, satisfying (1.3). Now given $\varepsilon > 0$, define the lengths $L(\varepsilon)^{\pm} \in (0,\infty)$ by

$$\|u^{\pm}_{1,\varphi,E}\|_{L(\varepsilon)^{\pm}} \|u^{\pm}_{2,\varphi,E}\|_{L(\varepsilon)^{\pm}} = \frac{1}{2\varepsilon} \ . \tag{2.3}$$

By the constancy of the Wronskian (namely, $W[u^{\pm}_{1,\varphi,E},\ u^{\pm}_{2,\varphi,E}] = 1$), at most one of the solutions $u^{\pm}_{1,\varphi,E}$, $u^{\pm}_{2,\varphi,E}$ belongs to $l^2(\mathbb{Z}^{\pm})$, the functions $L(\varepsilon)$ is well defined by (2.3) and $L(\varepsilon) \to \infty$ as $\varepsilon \to 0$ (see [15]).

As a consequence of Jitomirskaya-Last inequality (Theorem 1.1 of [14]), we have the following results that connects Hausdorff and packing continuity of the spectral measure of H to the scaling behavior of the (generalized) eigenfunctions of H:

Theorem 2.1. *(Theorem 1.2 in [14] and part of Theorem 14 in [3]) Let H be defined by the action (1.1) in $l^2(\mathbb{Z}^+)$, and let μ denote the spectral measure of H associated with the cyclic vector δ_0. Let $E \in \mathbb{R}$ and $\alpha \in (0,1)$. Then, for any $\varphi \in (-\pi/2, \pi/2]$, $(\overline{D}^{\alpha}\mu)(E) = \infty$ holds if, and only if,*

$$\liminf_{L\to\infty} \frac{\|u_{1,\varphi,E}\|_L}{\|u_{2,\varphi,E}\|_L^{\alpha/(2-\alpha)}} = 0,$$

and $(\underline{D}^{\alpha}\mu)(E) < \infty$ holds if

$$\limsup_{L\to\infty} \frac{\|u_{1,\varphi,E}\|_L}{\|u_{2,\varphi,E}\|_L^{\alpha/(2-\alpha)}} > 0.$$

Theorem 2.1 provides a tool for the analysis of dimensional properties of some spectral measures of Schrödinger operators.

Lemma 2.2. *Pick $E \in \sigma(H)$, and suppose that there exists a sequence $L_j \to \infty$ and every solution to $(H-E)u = 0$ with NIC obeys the estimate*

$$C_1 L_j^{\tau_1} \le \|u\|_{L_j} \le C_2 L_j^{\tau_2},$$

where C_1, C_2, τ_1, τ_2 are positive constants. Then, there exist a positive constant C_3 and a sequence $\varepsilon_j \to 0$ such that, for $\alpha = 2\tau_1/(\tau_1+\tau_2)$,

$$|m(E+i\varepsilon_j)| = \left| \frac{m^+(E+i\varepsilon_j)m^-(E+i\varepsilon_j) - 1}{m^+(E+i\varepsilon_j) + m^-(E+i\varepsilon_j)} \right| \le C_3\, \varepsilon_j^{\alpha-1}. \tag{2.4}$$

Consequently, μ is α-packing continuous.

Proof. The proof of Lemma 2.2 traces the same steps of the proof of Theorem 4 and Corollary 2.1 in [7], with simple adaptations. We conclude from Theorem 2.1 and (2.2) that μ is α-packing continuous. □

Proof. (Theorem 1.1) It follows, from hypothesis (1.5) and Lemma 2.2, that μ is α-packing continuous. The result is a consequence of the fact that $\mu_\phi \ll \mu$. □

3. Spectral packing continuity for Sturmian operators

In this section, we present the proof of Theorem 1.5, but first we recall some basic properties of the Sturmian potentials. Let us fix a rotation number θ and let (a_n) be the sequence of coefficients in its continued fraction expansion (1.6). Define the words S_n over the alphabet $\mathcal{A} = \{0, \lambda\}$ (with $0 \neq \lambda \in \mathbb{R}$ fixed) by

$$S_{-1} = \lambda, \quad S_0 = 0, \quad S_1 = S_0^{a_1-1} S_{-1}, \quad S_n = S_{n-1}^{a_n} S_{n-2}, \quad n \geq 2. \tag{3.1}$$

In particular, the word S_n has length q_n for each $n \geq 0$. For each potential of the form $V_{\lambda,\theta,\rho}(n) = \lambda \chi_{[1-\theta,1)}(n\theta + \rho \bmod 1)$, with $0 \neq \lambda \in \mathbb{R}$, $\theta \in [0,1)$ an irrational number and $\rho \in [0,1)$, it is possible to select a sequence a_n so that the potential is recovered through (3.1) (see [2, 7, 8] for more details).

Fix $E \in \mathbb{R}$; then, for each $w = w_1 \dots w_n \in \mathcal{A}^n$, the transfer matrix $M(E,w)$ is defined as

$$M(E,w) = \begin{pmatrix} E - w_n & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} E - w_1 & -1 \\ 1 & 0 \end{pmatrix}.$$

If u is a solution to (1.2), one has, for every $n \in \mathbb{N}$,

$$U(n+1) = M(E, V_{\lambda,\theta,\rho}(1) \dots V_{\lambda,\theta,\rho}(n)) U(1), \tag{3.2}$$

where

$$U(n) = \begin{pmatrix} u(n+1) \\ u(n) \end{pmatrix}.$$

Observe that the behavior of $\|u\|_L$, for L large, can be investigated through

$$\|U\|_L = \left(\sum_{n=1}^{[L]} \|U(n)\|^2 + (L - [L]) \|U([L]+1)\|^2 \right)^{\frac{1}{2}},$$

with $\|U(n)\|^2 = |u(n)|^2 + |u(n-1)|^2$. A simple calculation leads to

$$\frac{1}{2} \|U\|_L^2 \leq \|u\|_L^2 \leq \|U\|_L^2 \ .$$

Since the spectrum of $H_{\lambda,\theta,\rho}$ is independent of ρ [2], we just denote it by $\sigma(H_{\lambda,\theta})$. Put $x_n = \mathrm{tr}(M(E, S_{n-1}))$, $y_n = \mathrm{tr}(M(E, S_n))$ and $z_n = \mathrm{tr}(M(E, S_n S_{n-1}))$, with the explicit dependence on λ and E suppressed. According to results in [2, 7], for every $0 \neq \lambda \in \mathbb{R}$ there exists a $C_\lambda > 1$ such that

$$\max_n \{|x_n|, |y_n|, |z_n|\} \leq C_\lambda,$$

uniformly in $E \in \sigma(H_{\lambda,\theta})$ and every irrational θ. We emphasize that this property is important to obtain lower bounds for the solutions u to (1.2).

3.1. **Lower bound for the solutions.** In order to prove Theorem 1.5, we need to obtain lower bounds for all solutions to $(H_{\lambda,\theta,\rho} - E)u = 0$ with NIC and corresponding to energies $E \in \sigma(H_{\lambda,\theta})$. Thus, we will prove the following

Proposition 3.1. *Suppose that θ is an irrational number of quasibounded density. Then, for every $\lambda > 0$, there exists positive constants τ_1, C_1 and a sequence $(n_j)_{j\in\mathbb{N}}$ such that, for every solution u to (1.2) with NIC and corresponding to energies $E \in \sigma(H_{\lambda,\theta})$, one has*

$$\|u\|_{q_{n_j}} \geq C_1 q_{n_j}^{\tau_1}.$$

In the proof of Proposition 3.1, we will use the ideas of [5, 7] with obvious adaptations for a sequence.

Lemma 3.2. *(Lemma 4.1 in [7]) Let λ, θ, ρ be arbitrary, $E \in \sigma(H_{\lambda,\theta})$ and u a solution to (1.2) with NIC. Then, for every $n \geq 8$, the inequality*

$$\|U\|_{q_n} \geq D_\lambda \|U\|_{q_{n-8}}$$

holds true, where $D_\lambda = \left(1 + \frac{1}{4C_\lambda^2}\right)^{\frac{1}{2}}$.

Lemma 3.3. *Suppose that θ is an irrational number of quasibounded density. Then, there exist a constant C_θ and a sequence $(n_j)_{j\in\mathbb{N}}$ such that $q_{n_j} \leq C_\theta^{n_j}$.*

Proof. The proof of Lemma 3.3 traces the same steps of the proof of Lemma 2.3 in [5], with the obvious adaptations for a subsequence. More precisely, we consider the sequence $(r_n)_{n\in\mathbb{N}}$ defined recursively by

$$r_{n+1} = 2a_{n+1} r_n, \qquad n \in \mathbb{N},$$

with initial condition $r_1 = 2a_1$.

Note that $q_n \leq r_n$ and $r_n = \prod_{i=1}^{n} 2a_i$, for all $n \in \mathbb{N}$. Now, since θ is an irrational number of quasibounded density, there exist a constant B_θ and a sequence $(n_j)_{j\in\mathbb{N}}$ such that

$$\frac{1}{n_j}\sum_{i=1}^{n_j} 2a_i \leq B_\theta. \tag{3.3}$$

Thus, we obtain

$$\ln(q_{n_j})^{1/n_j} \leq \ln(r_{n_j})^{1/n_j} = \frac{1}{n_j}\sum_{i=1}^{n_j} \ln(2a_i) \leq B_\theta,$$

and consequently, $q_{n_j} \leq C_\theta^{n_j}$, with $C_\theta = e^{B_\theta}$. □

Proof. (Proposition 3.1) We have by Lemma 3.2, that for all $n_j \geq 8$,

$$\begin{aligned}\|U\|_{q_{n_j}} &\geq D_\lambda \|U\|_{q_{n_j-8}} \geq \ldots \geq D_\lambda^{[n_j/8]} \|U\|_{q_{n_j - 8[n_j/8]}} \\ &\geq D_\lambda^{[n_j/8]} \|U\|_{q_0} \geq D_\lambda^{(n_j/8)-1},\end{aligned}$$

with $D_\lambda > 1$ and $[n_j/8]$ is the integral part of $n_j/8$.

Thus, it follows from Lemma 3.3 the existence of a sequence $(n_j)_{j\in\mathbb{N}}$ such that $q_{n_j} \leq C_\theta^{n_j}$. Then, choose $\tau_1 > 0$, satisfying $C_\theta^{8\tau_1} \leq D_\lambda$,

$$\frac{\|U\|_{q_{n_j}}}{q_{n_j}^{\tau_1}} \geq \frac{D_\lambda^{(n_j/8)-1}}{C_\theta^{n_j \tau_1}} = \frac{1}{D_\lambda}\left(\frac{D_\lambda^{1/8}}{C_\theta^{\tau_1}}\right)^{n_j} \geq \frac{1}{D_\lambda},$$

which implies that $\|U\|_{q_{n_j}} \geq D_\lambda^{-1} q_{n_j}^{\tau_1}$. Therefore,

$$\|u\|_{q_{n_j}} \geq C_1 q_{n_j}^{\tau_1},$$

with $C_1 = 1/D_\lambda\sqrt{2}$. □

3.2. **Upper bound for the solutions.** Now, we obtain power-law upper bounds for the solutions to (1.2). We prove the following

Proposition 3.4. *Suppose that θ is an irrational number of quasibounded density. Then for every $\lambda > 0$, there exists positive constants τ_2, C_2 and a sequence $(n_j)_{j\in\mathbb{N}}$ such that, for every solution u to (1.2), with NIC and corresponding to energies $E \in \sigma(H_{\lambda,\theta})$, one has*

$$\|u\|_{q_{n_j}} \leq C_2 q_{n_j}^{\tau_2}.$$

We will use, in the proof of Proposition 3.4, ideas employed in [12, 13] which produces estimates for the norm of the transfer matrices associated with the operator $H_{\lambda,\theta,\rho}$. In order to avoid cumbersome notations, we set

$$M(m) := M(E, V_{\lambda,\theta,\rho}(1)\ldots V_{\lambda,\theta,\rho}(m)).$$

Lemma 3.5. *(Theorem 9 in [12]) For any integer m, written as $m = \sum_{i=0}^{n} \epsilon_i q_i$, with all ϵ_i integers, one has*

$$\|M(m)\| \leq J_1^{\sum_{i=1}^{n+1} a_i} J_2^{\sum_{i=0}^{n} \epsilon_i},$$

where J_1 and J_2 are positive constants such that $J_1 \geq J_2$.

Lemma 3.6. *Let $(q_{n_j})_{j\in\mathbb{N}}$ be the sequence obtained in Lemma 3.3, and let m be a positive integer with $m < q_{n_j}$, for some n_j. Then, one has the expansion*

$$m = \sum_{i=0}^{n_j-1} \epsilon_i q_i, \tag{3.4}$$

with $0 \leq \epsilon_i \leq a_{i+1}$, $i = 0, 1, \ldots, n_j - 1$.

Proof. For the sequence $(q_{n_j})_{j\in\mathbb{N}}$, we can consider, without loss of generality, that $q_{n_0} = q_0 = 1$ and $q_{n_1} = q_1 = a_1$. We prove the result by induction on $j \in \mathbb{N}$. If $q_{n_0} = 1 \leq m < q_{n_1} = a_1$, then $m = \epsilon_0 q_0 = \epsilon_0$, with $1 \leq \epsilon_0 \leq a_1$. Suppose that the result is valid for $m < q_{n_{j-1}}$.

Now, suppose that $q_{n_{j-1}} \leq m < q_{n_j}$. We will analyze all possible values that m may assume in the interval $[q_{n_{j-1}}, q_{n_j})$.

For $q_{n_{j-1}} \leq m < q_{n_{j-1}+1}$, write $\epsilon_{n_{j-1}} = \left[\frac{m}{q_{n_{j-1}}}\right]$. Then, $m - \epsilon_{n_{j-1}} q_{n_{j-1}} < q_{n_{j-1}}$ and

$$m - \epsilon_{n_{j-1}} q_{n_{j-1}} = \sum_{i=0}^{n_{j-1}-1} \epsilon_i q_i,$$

for $\epsilon_i \leq a_{i+1}$, $i = 0, \ldots, n_{j-1} - 1$. We also have

$$\epsilon_{n_{j-1}} < \left[\frac{q_{n_{j-1}+1}}{q_{n_{j-1}}}\right] = \left[\frac{a_{n_{j-1}+1} q_{n_{j-1}} + q_{n_{j-1}-1}}{q_{n_{j-1}}}\right] = a_{n_{j-1}+1},$$

and $\epsilon_i = 0$ for $i = n_{j-1} + 1, \ldots, n_j - 1$.

The next step is to consider the case $q_{n_{j-1}+1} \leq m < q_{n_{j-1}+2}$, which follows from the same considerations of the previous case. Thus, proceeding inductively with this analysis on the values of m, we obtain (3.4) for $q_{n_j-1} \leq m < q_{n_j}$, that is,

$$m - \epsilon_{n_j-1} q_{n_j-1} = \sum_{i=0}^{n_j-2} \epsilon_i q_i,$$

where $\epsilon_i \leq a_{i+1}$ for $i = 0, \ldots, n_j - 2$ and $\epsilon_{n_j-1} = \left[\frac{m}{q_{n_j-1}}\right] < a_{n_j}$. □

Proof. (Proposition 3.4) If u is a solution to (1.2) with NIC, one has, by (3.2), that $|u(m)| \leq \|M(m)\|$. Then,

$$\begin{aligned} \|u\|^2_{q_{n_j}} &= \sum_{m=1}^{q_{n_j}} |u(m)|^2 \leq \sum_{m=1}^{q_{n_j}} \|M(m)\|^2 \\ &\leq q_{n_j} (J_1)^{4\sum_{i=1}^{n_j} a_i} \leq q_{n_j} (J_1)^{2B_\theta n_j} \\ &\leq q_{n_j}^{2\tau_2}, \end{aligned}$$

with $\tau_2 \geq \frac{1}{2} + \ln(J_1)$, where we have used Lemmas 3.5 and 3.6 in the second inequality, (3.3) in the third, and Lemma 3.3 in the last one. □

Proof. (Theorem 1.5) Let θ be an irrational number of quasibounded density. Thus, by Propositions 3.1 and 3.4, there exists a sequence $(q_{n_j})_{j\in\mathbb{N}}$ such that for every $\lambda > 0$, there exist τ_1, τ_2 and constants C_1, C_2, such that, for any solution to (1.2) with NIC, one has

$$C_1 q_{n_j}^{\tau_1} \leq \|u\|_{q_{n_j}} \leq C_2 q_{n_j}^{\tau_2}.$$

Therefore, by Theorem 1.1, the spectrum of $H_{\lambda,\theta,\rho}$ is purely α-packing continuous, with $\alpha = 2\tau_1/\tau_1 + \tau_2$. □

Remark 3.7. We have, as a particular case of Theorem 1.5, that if θ is an irrational number of bounded density, then the spectral measure of the operator $H_{\lambda,\theta,\rho}$ is purely α_P-packing continuous with $\alpha_P = \frac{2\tau_1}{\tau_1+\tau_2}$, where $\tau_1, \tau_2 > 0$ are of the form

$$C_1 q_n^{\tau_1} \leq \|u\|_{q_n} \leq C_2 q_n^{\tau_2},$$

for any solution to (1.2) with NIC. However, it is well known [5, 7, 12, 15] that if θ is an irrational number of bounded density, then $H_{\lambda,\theta,\rho}$ has purely α_H-Hausdorff continuous spectrum for $\alpha_H = \frac{2\gamma_1}{\gamma_1+\gamma_2}$, where $\gamma_1, \gamma_2 > 0$ satisfy the relation (1.4).

Due to the way τ_1 and τ_2 were obtained in Propositions 3.1 and 3.4, we note that these estimates have important relations with γ_1, γ_2. More specifically, we observe that $\tau_1 > \gamma_1$ and $\tau_2 < \gamma_2$; consequently, it follows that $\alpha_P > \alpha_H$ from such estimates.

This observation is verified rewriting the proof of Proposition 2.1 in [5], which has $\gamma_1 \equiv \tau_1 - \varepsilon$, with $\varepsilon \in \left(\frac{\ln C_{\theta,1} - \ln C_{\theta,2}}{\ln C_{\theta,1}} \tau_1, \tau_1\right)$ and $C^n_{\theta,2} \leq q_n \leq C^n_{\theta,1}$.

And the estimate for γ_2 was obtained by Corollary 10 in [12], where one has to $\gamma_2 \geq \frac{1}{2} + \frac{4}{\ln 2} \ln J_1$, with J_1 as in Lemma 3.5 above. Therefore,

$$\gamma_2 \geq \frac{1}{2} + \frac{4}{\ln 2} \ln J_1 > \frac{1}{2} + \ln J_1 = \tau_2.$$

Remark 3.8. We present an example of irrational number $\theta = [0; a_1, a_2, \ldots]$ of quasibounded density that is not of bounded density. Let

$$A_n = \frac{1}{n}\sum_{i=1}^{n} a_i;$$

it is enough to build (a_n) and a subsequence of indices $(n_j)_{j\in\mathbb{N}}$ such that $A_{n_j} \leq 2$ and $A_{n_j+1} \geq j$, for all j.

For this write $n_1 = 2$ and take $a_1 = a_2 = 1$, $a_3 = 3$; choose $n_2 = 5$ with $a_4 = a_5 = 1$, $a_6 = 2 * 6$; now consider $n_3 = 6A_6 = 19$ and take $a_7 = \ldots = a_{19}$=1, $a_{20} = 3 * 20$. Proceeding this way, we obtain the subsequence $(n_j)_{j\in\mathbb{N}}$ defined recursively by

$$n_{j+1} = A_{n_j+1}(n_j+1), \quad \forall j > 2,$$

with the terms (a_n) given by

$$a_n = \begin{cases} j(n_j+1), & n \in J \\ 1, & n \notin J \end{cases},$$

where $J = \{n_j + 1 : j \in \mathbb{N}\}$.

4. Stability of spectral packing dimension

We present in this section stability results of spectral packing dimensional properties for some discrete Schrödinger operators of the form (1.1) under suitable (real) polynomially decaying perturbations $P = \{P(n)\}$, that is, when V is replaced by $V + P$. The results obtained here are analogous to the results presented in [1] for the Hausdorff dimensional setting. As in [1], we are interested in energies in the set

$$S(H) := \{E \mid \exists\,\varphi \text{ s.t. } u_{1,\varphi,E}$$
$$\text{is a subordinate solution to (1.2) and } u_{1,\varphi,E} \notin l^2(\mathbb{Z}^+)\}\,.$$

It is know [18] that, for any $\varphi \in [-\pi/2, \pi/2)$, the singular continuous part of the spectral measure of H_φ is supported in $S(H)$. In case of whole-line operators, $S(H)$ should be defined as [10]

$$\{E \mid \exists \text{ a solution to (1.2) which is subordinate at both ends } \pm\infty$$
$$\text{and which is not in } l^2(\mathbb{Z})\}\,,$$

and the singular continuous parts of the spectral measures are supported in this set; note that if no solution to (1.2) satisfies such condition at one end, then the corresponding energy E does not belong to the singular continuous component.

We have the following

Theorem 4.1. *Let $E \in S(H)$ and $u_{1,\varphi,E}$, $u_{2,\varphi,E}$ be solutions to (1.2) satisfying (1.3). Suppose that there exist positive constants $C_1, C_2, \gamma_1, \gamma_2$ such that every solution to (1.2) with NIC obeys the estimates (1.4) for $L > 0$ sufficiently large. Suppose also that, for every $n \in \mathbb{N}$ and $p > 3\gamma_2 - \gamma_1$, there exists a positive constant C_3 such that*

$$|P(n)| \leq C_3(1+n)^{-p}. \tag{4.1}$$

Then, $E \in S(H+P)$, and for all $\kappa \in [0,1]$,

$$\limsup_{L\to\infty} \frac{\|u_{1,\varphi,E}\|_L}{\|u_{2,\varphi,E}\|_L^{\kappa}} = \limsup_{L\to\infty} \frac{\|v_{1,\tilde\varphi,E}\|_L}{\|v_{2,\tilde\varphi,E}\|_L^{\kappa}}, \tag{4.2}$$

where $v_{1,\tilde{\varphi},E}$ is the solution to (1.2) for $H+P$, which satisfies the initial condition (1.3) with some phase $\tilde{\varphi}$, and $v_{2,\tilde{\varphi},E}$ satisfying the corresponding orthogonal condition (always for the operator $H+P$).

Proof. The proof of this theorem is analogous to the proof of Theorem 1.3 in [1], with simple modification to the upper limit. □

We emphasize that condition (1.4) is essential in Theorem 4.1 (i.e., bounds like in (1.5) are not enough for the result). Therefore, we have considered, in Corollary 1.6, Schrödinger operators with Sturmian potentials whose rotation number is of bounded density.

Proof. (Corollary 1.6) The proof of Corollary 1.6 is analogous to the proof of Theorem 1.1 in [1]. More specifically, we have, by [12, 5, 7], that for Schrödinger operators with Sturmian potentials whose rotation numbers are of bounded density, there exist power-law bounds of the form (1.4) for every solution u to (1.2) (with NIC). We have, by Theorem 1.5, that if θ is of bounded density (Remark 3.7), then for every $\lambda \neq 0$, there exists $\alpha = \alpha(\lambda, \theta) > 0$ such that for every $\rho \in [0, 1)$, the spectral measure $H_{\lambda,\theta,\rho}$ is purely α-packing continuous, with $\alpha = \frac{2\tau_1}{\tau_1+\tau_2}$.

We note again [5, 7] that if one is able to establish uniform power-law bounds on the restriction of the operator to the right half-line, then the resulting α-continuity is independent of the potential on the left half-line.

Suppose that $\sigma(H^P_{\lambda,\theta,\rho})$ has some singular continuous component; now, since the perturbation decays as $|P(n)| \leq C(1+|n|)^{-p}$, with $p > 3\gamma_2 - \gamma_1$, it is a compact perturbation and the essential spectrum is preserved. Thus, this singular continuous component is supported in $S(H_{\lambda,\theta,\rho})$ and, by Theorem 4.1, we obtain that the asymptotic behavior of generalized eigenfunctions of the operators $H^P_{\lambda,\theta,\rho}$ (that is, the solutions to (1.2) in (1.7) is the same of the eigenfunctions of the unperturbed operators $H_{\lambda,\theta,\rho}$; and again, by the α-subordinacy theory (Theorem 2.1), such component is still α-packing continuous for these perturbed operators, with $\alpha = \frac{2\tau_1}{\tau_1+\tau_2}$. □

According to results in [1], we can also apply Theorem 4.1 to operators with sparse potentials. We reconsider here the class of operators H^α_φ [3, 14, 23] defined by the action (1.1) in $l^2(\mathbb{Z}^+)$, along with a phase boundary condition

$$\psi(0)\cos\varphi + \psi(1)\sin\varphi = 0, \quad \varphi \in (-\pi/2, \pi/2], \tag{4.3}$$

and, for each $\alpha \in (0,1)$, sparse potentials

$$V(n) = \begin{cases} x_j^{(1-\alpha)/2\alpha}, & n = x_j \in \mathcal{B} \\ 0, & n \notin \mathcal{B} \end{cases}, \tag{4.4}$$

where $\mathcal{B} = (x_j)_j = \left(2^{j^j}\right)_j$. It is known that the restriction of its spectral measure to the interval $(-2,2)$ is one packing dimensional [3, 4] for every boundary phase φ.

Theorem 4.2. *Fix $\alpha \in (0,1)$. Let H^α_φ be as above and*

$$(H^{P,\alpha}_\varphi \psi)(n) := (H^\alpha_\varphi \psi)(n) + P(n)\psi(n), \quad \psi \in l^2(\mathbb{Z}^+), \tag{4.5}$$

with $|P(n)| \leq C(1+n)^{-p}$ for all n and some $C > 0$, $p > \min\{3/(2\alpha), (2-\alpha)/\alpha\}$. Then, the restriction of the spectral measure of the operator $H^{P,\alpha}_\varphi$ to $(-2,2)$ is also one-packing dimensional, for all boundary phase $\varphi \in (-\pi/2, \pi/2]$.

Proof. The proof of Theorem 4.2 follows the same steps of the proof of Theorem 1.2 in [1]. □

Appendix: Hausdorff and packing dimensions

We recall in this section some definitions and concepts regarding Hausdorff and packing measures, also to fix notation. Most of the material exposed here is based on [3, 9, 14, 19, 20, 21].

Definition 4.3. Given a set $S \subset \mathbb{R}$ and $\alpha \in [0,1]$, consider the number

$$Q_{\alpha,\delta}(S) = \inf\left\{ \sum_{k=1}^{\infty} |I_k|^\alpha \ \mid \ |I_k| < \delta, \ \forall k; \ S \subset \bigcup_{k=1}^{\infty} I_k \right\},$$

with the infimum taken over all covers of S by intervals I_k of size at most δ. The limit

$$h^\alpha(S) = \lim_{\delta \to 0} Q_{\alpha,\delta}(S)$$

is called the α-dimensional Hausdorff measure of S.

The α-dimensional Hausdorff measure, h^α, is an outer measure on subsets of $\mathbb{R}$ [21]. It is known that for every set S, there is a unique α_S such that $h^\alpha(S) = 0$ if $\alpha > \alpha_S$ and $h^\alpha(S) = \infty$ if $\alpha_S < \alpha$. The number α_S is called the Hausdorff dimension of the set S, usually denoted by $\dim_H(S)$. Particular examples of h^α are the counting measure for $\alpha = 0$ and the Lebesgue measure for $\alpha = 1$.

Now the definition of packing measure. A δ-packing of an arbitrary set $S \subset \mathbb{R}$ is a countable disjoint collection $(B(x_k, r_k))_{k\in\mathbb{N}}$ of closed intervals centered at $x_k \in S$ with radius $r_k \le \delta/2$. The (α,δ)-premeasure $P_\delta^\alpha(S)$ is defined by

$$P_\delta^\alpha(S) = \sup\left\{ \sum_{k=1}^{\infty} (2r_k)^\alpha : (B(x_k, r_k))_{k\in\mathbb{N}} \text{ is a } \delta\text{-packing of } S \right\}$$

the supremum taken over all $\delta-$ packings of S.

Definition 4.4. The α-packing measure $P^\alpha(S)$ of S is constructed by a procedure in two steps: first, take the decreasing limit

$$\underline{P}^\alpha(S) = \lim_{\delta \to 0} P_\delta^\alpha(S)$$

and then

$$P^\alpha(S) = \inf\left\{ \sum_{k=1}^{\infty} \underline{P}^\alpha(S_k) : S \subset \cup_{k=1}^{\infty} S_k, \ S_k \ \text{ disjoint Borel sets} \right\}.$$

It follows, by Definition 4.4, that $P^\alpha(S)$ is an outer measure on $\mathbb{R}$. The so-called packing dimension of the set S, denoted by $\dim_P(S)$, is defined as the infimum of all α such that $P^\alpha(S) = 0$, which coincides with the supremum of all α with $P^\alpha(S) = \infty$. It is possible to show (see [9]) that the Hausdorff and packing dimensions are related by the inequality $\dim_H(S) \le \dim_P(S)$.

Definition 4.5. Let μ be a Borel measure in $\mathbb{R}$ and $\alpha \in [0,1]$.

- **: (i)** μ is called α-Hausdorff (α-packing) continuous if $\mu(S) = 0$ for every Borel set S with $h^\alpha(S) = 0$ (resp. $P^\alpha(S) = 0$).
- **: (ii)** μ is called α-Hausdorff (α-packing) singular if μ is supported on some Borel set S, i.e., $\mu(\mathbb{R}\backslash S) = 0$ with $h^\alpha(S) = 0$ (resp. $P^\alpha(S) = 0$).

Definition 4.6. A Borel measure μ in $\mathbb{R}$ is said to have exact Hausdorff (packing) dimension α, for some $\alpha \in (0,1)$, and denoted by $\dim_H(\mu)$ (resp.$\dim_P(\mu)$), if two requirements hold:

: **(i)** for every set S with $\dim_H(S) < \alpha$ (resp. $\dim_P(S) < \alpha$), one has $\mu(S) = 0$;
: **(ii)** there is a Borel set, S_0, of Hausdorff (resp. packing) dimension α which supports μ.

A Borel measure μ in $\mathbb{R}$ is said to be 0-Hausdorff (0-packing) dimensional if it is supported on a set with $\dim_H(S) = 0$ (resp. $\dim_P(S) = 0$) and 1-Hausdorff (1-packing) dimensional if $\mu(S) = 0$ for any set S with $\dim_H(S) < 1$ (resp. $\dim_P(S) < 1$).

Remark 4.7. According to Definitions 4.5 and 4.6, a Borel measure μ in $\mathbb{R}$ is of exact Hausdorff (packing) dimension α if, for every $\varepsilon > 0$, it is simultaneously $(\alpha-\varepsilon)$-Hausdorff (resp. packing) continuous and $(\alpha+\varepsilon)$- Hausdorff (resp. packing) singular.

Given a finite Borel measure μ and $\alpha \in [0,1]$, write

$$(\overline{D}^{\alpha}\mu)(E) := \limsup_{\varepsilon\to 0} \frac{\mu((E-\varepsilon, E+\varepsilon))}{(2\varepsilon)^{\alpha}} \quad \text{and} \quad (\underline{D}^{\alpha}\mu)(E) := \liminf_{\varepsilon\to 0} \frac{\mu((E-\varepsilon, E+\varepsilon))}{(2\varepsilon)^{\alpha}}.$$

Theorem 4.8. *Let $\alpha \in [0,1]$ and μ a Borel measure on $\mathbb{R}$, denote*

$$T^{\alpha}_{\infty} = \{E \in \mathbb{R} : (\overline{D}^{\alpha}\mu)(E) = \infty\}, \quad U^{\alpha}_{\infty} = \{E \in \mathbb{R} : (\underline{D}^{\alpha}\mu)(E) = \infty\}.$$

Then T^{α}_{∞} and U^{α}_{∞} are Borel sets, and

(1) $h^{\alpha}(T^{\alpha}_{\infty}) = 0$.
(2) $P^{\alpha}(U^{\alpha}_{\infty}) = 0$.
(3) $\mu(S \cap (\mathbb{R}\backslash T^{\alpha}_{\infty})) = 0$, for any S with $h^{\alpha}(S) = 0$.
(4) $\mu(S \cap (\mathbb{R}\backslash U^{\alpha}_{\infty})) = 0$, for any S with $P^{\alpha}(S) = 0$.

Proof. Items (1) and (3) are well known and proven in Chapter 3 in [21], and the proofs of items (2) and (4) are in [3]. □

Remark 4.9. The restriction $\mu_{\alpha Hs} := \mu(T^{\alpha}_{\infty} \cap \cdot)$ is α-Hausdorff singular, $\mu_{\alpha Ps} := \mu(U^{\alpha}_{\infty} \cap \cdot)$ is α-packing singular; and $\mu_{\alpha Hc} := \mu((\mathbb{R}\backslash T^{\alpha}_{\infty}) \cap \cdot)$ is α-Hausdorff continuous, $\mu_{\alpha Pc} := \mu((\mathbb{R}\backslash T^{\alpha}_{\infty}) \cap \cdot)$ is α-packing continuous. Thus, each measure decomposes uniquely into an α-Hausdorff (packing) continuous part and an α-Hausdorff (packing) singular part: $\mu = \mu_{\alpha Hs} + \mu_{\alpha Hc}$ (resp. $\mu = \mu_{\alpha Ps} + \mu_{\alpha Pc}$).

Moreover, an α-Hausdorff (packing) singular measure is such that $(\overline{D}^{\alpha}\mu)(E) = \infty$ (resp. $(\underline{D}^{\alpha}\mu)(E) = \infty$) a.e (with respect to it), while an α-Hausdorff (packing) continuous measure is such that $(\overline{D}^{\alpha}\mu)(E) < \infty$ (resp. $(\underline{D}^{\alpha}\mu)(E) < \infty$) a.e (see Chapter 3 in [21] and [3]).

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