On the Neumann Laplacian in nonuniformly collapsing strips

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March 1, 2017

Abstract

Consider the Neumann Laplacian in the region below the graph of $\varepsilon g(x)$, for a positive smooth function $g : [a, \infty) \to \mathbb{R}$ with both $g'(x)/g(x)$ and $(g'(x)/g(x))'$ bounded. As $\varepsilon \to 0$ such region collapses to $[a, \infty)$ and an effective operator is found, which has Robin boundary conditions at a. Then we recover (under suitable assumptions in the case of unbounded g) such effective operators through uniformly collapsing regions; in such approach, we have (roughly) got norm resolvent convergence for g diverging less than exponentially.

Keywords collapsing strips, thick regions, Laplacian, resolvent convergence, reduction of dimension.

Mathematics Subject Classification 2010: 35B40, 35B25, 35K57, 81Q15.

1 Introduction and results

Domain collapsing and dimensional reduction, including the confinement of quantum particles, have been studied in various aspects [2, 3, 5, 6, 7, 8, 9, 10, 11]. For example, considering the confinement in a determined region, a question of great interest is to find an effective operator when some directions are squeezed. In this process the effective operator can be influenced by the geometry of the region (as planar strips or tubes in the space), as well as the boundary condition at its border (the most usual ones are Dirichlet and Neumann conditions). It is also worth mentioning works [1] on the Neumann problem in thin domains with a very high oscillatory behavior. Usually the confining process is supposed uniform along the squeezed directions. Here we consider effective operators for the Neumann Laplacian in some nonuniformly confining planar processes and propose uniform approximations for them; this possibility is rather surprising, since the original problem has unbounded cross sections during the whole process whereas we have got the same effective operators through suitable chosen regions with uniformly vanishing cross sections!

Consider the situation of an initial region given by the subset of points of $\mathbb{R}^2$ between the x-axis and the graph of a positive and smooth function $g : I \to \mathbb{R}$, with I being an

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interval of $\mathbb{R}$. For each $\varepsilon > 0$, consider the set $\Omega_\varepsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < \varepsilon\, g(x), x \in I\}$ and the Laplace operator restricted to it. At the boundary $\partial\Omega_\varepsilon$, take the Neumann condition. A question is to know in what sense the limit of this problem exists as $\varepsilon \to 0$ (collapsing domain to I). Hale and Raugel [9] have obtained some results under the assumption that g is a bounded function (and I a bounded interval). One of the goals of this work is to find, under some additional conditions on g, effective operators (in the strong resolvent sense) also for some unbounded functions g, for which Ω_ε does not collapse uniformly as $\varepsilon \to 0$ (a situation we call *thick strips* and, to our knowledge, it has not been considered up to now); see Theorem 1 ahead.

Another goal here is to approximate a divergent g by a sequence of bounded functions $(g_\varepsilon)_\varepsilon$ with pointwise convergence $g_\varepsilon \to g$, as illustrated in Figure 1, but so that the squeezing regions, defined through $\varepsilon g_\varepsilon$, collapse uniformly as $\varepsilon \to 0$. In some cases we will actually find the same effective operators as before, but the resolvent convergence will be uniformly if (roughly) g diverges less than exponentially at infinity; see Theorems 3 and 4, as well as explicitly examples in Classes I and II ahead.

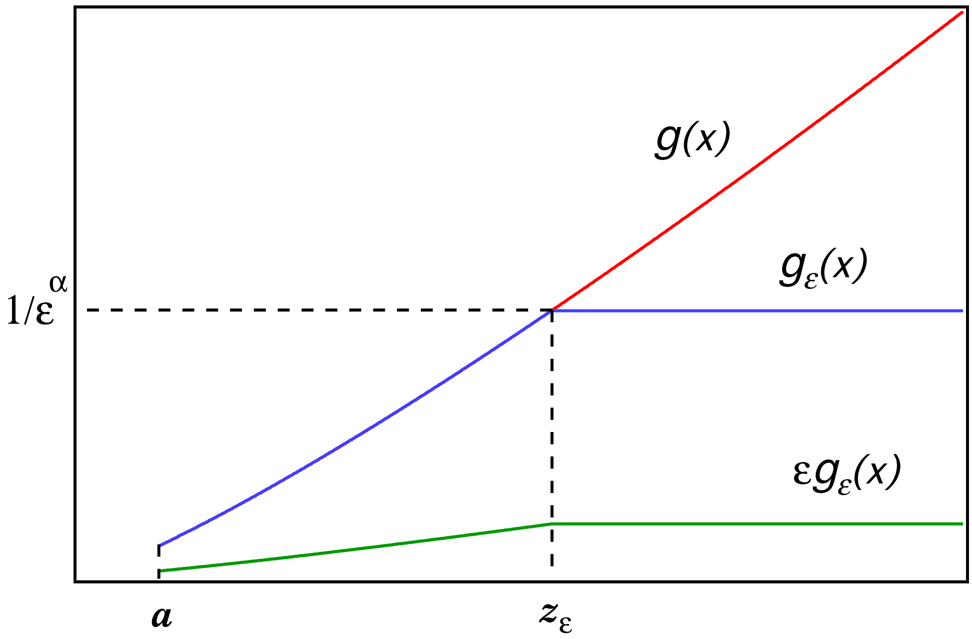

Figure 1: Illustration of the pointwise approximation of the unbounded function g with the sequence g_ε, and the uniformly collapsing $\varepsilon g_\varepsilon$ (for large x, the general behavior of εg is the same as g).

1.1 Collapsing thick strips

In this and next subsections, we describe in more details the results of this work, while most proofs are left to other sections. For simplicity, we impose convenient conditions on g from scratch:

(i) $g : [a, \infty) \to \mathbb{R}$ is a C^2 function with $g(x) > r_0 > 0$ for all x; furthermore, $j(x) := g'(x)/(2g(x))$ and its derivative $j'(x)$ are supposed to be bounded (con-

tinuous) functions.

Let For each $\varepsilon > 0$, consider the region

$$\Lambda_\varepsilon := \{(x,y) \in \mathbb{R}^2 \mid 0 < y < \varepsilon g(x),\ x \in [a,\infty)\}$$

and the quadratic form

$$m_\varepsilon(v) = \int_{\Lambda_\varepsilon} |\nabla v|^2 \mathrm{d}x, \quad \mathrm{dom}\ m_\varepsilon = H^1(\Lambda_\varepsilon)\,,$$

associated with the Neumann Laplacian in Λ_ε. After some changes of variables, similar to those in Section 2 of this work, $m_\varepsilon(v)$ is cast as

$$n_\varepsilon(\varphi) := \int_\Lambda \left(\left| \varphi' - \frac{g'}{2g}\varphi - y\,\varphi_y \frac{g'}{g} \right|^2 + \frac{|\varphi_y|^2}{\varepsilon^2 g^2} \right) \mathrm{d}x\mathrm{d}y, \qquad \varphi \in \mathrm{dom}\ n_\varepsilon \subset H^1(\Lambda),$$

where $\Lambda := [a,\infty) \times (0,1)$ is a fixed region and φ_y denotes the derivative with respect to the variable y. Note that, as $\varepsilon \to 0$,

$$n_\varepsilon(\varphi) \ \longrightarrow \ \begin{cases} \int_\Lambda |\varphi' - j(x)\,\varphi|^2\,\mathrm{d}x, & \text{if} \quad \varphi_y = 0, \\ \infty, \quad \text{if} \quad \varphi_y \neq 0. \end{cases}$$

From this analysis, it is natural to introduce the subspace

$$\mathcal{J} := \{\varphi \in H^1(\Lambda) \mid \varphi(x,y) = w(x)1,\ w \in H^1([a,\infty))\}\,.$$

Note that $\mathcal{J}$ is directly related to the fact that the first eigenvalue of the Neumann Laplacian in a bounded region is zero (and the constant functions form the corresponding eigenspace). It is important to observe that, since $j(x)$ is bounded, $\mathcal{J} \subset \mathrm{dom}\ n_\varepsilon$ for all $0 < \varepsilon \leq 1$

Denote by S_ε the self-adjoint operator associated with $n_\varepsilon(\psi)$ and let $\mathcal{H}_0$ be the closure of $\mathcal{J}$ in $\mathrm{L}^2(\Lambda)$. Define the one-dimensional quadratic form

$$n(w) := \lim_{\varepsilon\to 0} n_\varepsilon(w1) = \int_a^\infty |w' - j(x)w|^2\,\mathrm{d}x, \quad \mathrm{dom}\ n = H^1([a,\infty)),$$

and denote by S the associated self-adjoint operator. By Theorem 1, S is the effective operator in this situation. We mention that very different effective operators are obtained if Dirichlet condition is considered; for instance, see the works by Friedlander and Solomyak [7, 8] which has studied the Dirichlet case and g with a unique global maximum.

Theorem 1. *The sequence S_ε converges in the strong resolvent sense to the self-adjoint operator S in $\mathcal{H}_0$. I.e., for all $f \in \mathrm{L}^2(\Lambda)$,*

$$\left\| S_\varepsilon^{-1} f - (S^{-1} \oplus 0) f \right\| \to 0,$$

as $\varepsilon \to 0$, where 0 denotes the null operator on $\mathcal{H}_0^\perp$.

Proof. For $0 < \varepsilon_2 < \varepsilon_1$, we have $0 \leq n_{\varepsilon_1}(\psi) \leq n_{\varepsilon_2}(\psi)$, for all $\psi \in H^1(\Lambda)$. Thus, the proof follows directly by Kato-Robinson Theorem (see Theorem 10.4.2 in [4]). □

An important point is to characterize the action and the domain of the effective operator S; since $j(x)$ and $j'(x)$ are supposed to be bounded functions, $\varrho(x) := j^2(x) + j'(x)$ is bounded as well.

Theorem 2. *If (i) holds, then*

$$(Sw)(x) := -w''(x) + \varrho(x)w(x), \tag{1}$$

with a Robin condition at the end point a, that is,

$$\mathrm{dom}\, S = \left\{ w \in H^2([a,\infty)) \mid j(a)\, w(a) = w'(a) \right\}.$$

A more detailed discussion about this characterization, from which the proof of Theorem 2 follows, is presented in the Appendix to this work. Selected explicitly examples will be presented later on in this section.

1.2 Uniformly collapsing regions

We pass now to details of the second goal of this work, that is, finding uniformly collapsing regions Q_ε whose effective operator coincides with S, actually a more delicate question. Let $\varepsilon_0 > 0$ and, for each $\varepsilon \in (0, \varepsilon_0)$, pick a positive sequence of bounded functions $g_\varepsilon : [a, +\infty) \to \mathbb{R}$ that converges pointwise to the unbounded g; this simulates $\{(x,y) \in \mathbb{R}^2 \mid 0 < y < g(x), x \in [a,\infty)\}$ being approximated by the regions $\{(x,y) \in \mathbb{R}^2 \mid 0 < y < g_\varepsilon(x), x \in [a,\infty)\}$; see Figure 1. Under appropriate conditions, we will have collapsing regions to $[a,\infty)$ by considering εg (nonuniformly) and $\varepsilon g_\varepsilon$ (uniformly).

Suppose that each g_ε is a continuous and smooth function by parts which is used to delimit the region

$$Q_\varepsilon := \left\{ (x,y) \in \mathbb{R}^2 \mid 0 < y < \varepsilon g_\varepsilon(x),\, x \in [a,\infty) \right\},$$

and the Neumann Laplacian restricted to Q_ε. For technical reasons, we pick $c > 0$ and consider the quadratic form

$$f_\varepsilon(\psi) = \int_{Q_\varepsilon} \left(|\nabla\psi|^2 + c|\psi|^2 \right) \mathrm{d}x\mathrm{d}y,$$

with $\mathrm{dom}\, f_\varepsilon = H^1(Q_\varepsilon)$. After some changes of variables (see Section 2 of this work), we pass to study the sequence

$$h_\varepsilon(\psi) = \int_Q \left(\left| \psi' - \frac{g_\varepsilon'}{2g_\varepsilon}\psi - y\frac{g_\varepsilon'}{g_\varepsilon}\psi_y \right|^2 + \frac{|\psi_y|^2}{\varepsilon^2 g_\varepsilon^2} + c\,|\psi|^2 \right) \mathrm{d}x\mathrm{d}y, \tag{2}$$

$\mathrm{dom}\, h_\varepsilon = H^1(Q) \subset \mathrm{L}^2(Q)$, where $Q := [a,\infty) \times (0,1)$. Since for each $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ so that, for all x, $C_\varepsilon^{-1} \le \varepsilon g_\varepsilon(x) \le C_\varepsilon$, the change of variables from Q_ε to Q is a global diffeomorphism and bi-Lipschitz, thus it defines an isomorphism between the Sobolev spaces $H^1(Q)$ and $H^1(Q_\varepsilon)$; so $\mathrm{dom}\, h_\varepsilon = H^1(Q)$.

Denote by H_ε the self-adjoint operator associated with $h_\varepsilon(\psi)$; this is the operator whose behavior we are interested in understanding as $\varepsilon \to 0$. Note that we are not guaranteed to apply Kato-Robinson Theorem in this situation, and we need to work alternatively.

Now, think of the subspace

$$\mathcal{L} := \left\{ w(x)\, 1 \mid w \in \mathrm{L}^2([a,\infty)) \right\} \subset \mathrm{L}^2(Q),$$

the one-dimensional quadratic form

$$t_\varepsilon(w) := h_\varepsilon(w\, 1) = \int_a^\infty \left(\left| w' - \frac{g_\varepsilon'}{2g_\varepsilon} w \right|^2 + c\,|w|^2 \right) \mathrm{d}x, \qquad \mathrm{dom}\, t_\varepsilon = H^1([a,\infty)), \tag{3}$$

and denote by T_ε the associated self-adjoint operator ($t_\varepsilon(w)$ is a closed form since it is the restriction of a closed form to a closed subspace of $H^1(Q)$).

Suppose that for $\varepsilon \in (0, \varepsilon_0)$ there exist $s(\varepsilon)$ and $r(\varepsilon)$ so that, for all $x \in [a, \infty)$:

(ii) $g_\varepsilon(x) \le s(\varepsilon)$, and $\varepsilon\, s(\varepsilon) \to 0$, as $\varepsilon \to 0$ (implying uniformly collapsing Q_ε);

(iii) $(g'_\varepsilon(x)/g_\varepsilon(x))^2 \le r(\varepsilon)$, and $\varepsilon\, s(\varepsilon)\, r(\varepsilon)^{1/2} \to 0$, as $\varepsilon \to 0$.

For simplicity, along this work we use the same symbol K to denote different positive constants. Our general result on dimensional reduction in this case is a norm resolvent approximation and states as follows.

Theorem 3. *Assume that conditions (i)-(iii) hold true; then, there exists $K > 0$, so that, for all $\varepsilon > 0$ small enough,*

$$\left\| H_\varepsilon^{-1} - \left(T_\varepsilon^{-1} \oplus 0\right)\right\| < \left(\frac{\varepsilon^2\, s^2(\varepsilon)}{\pi^2} + K\, \frac{\varepsilon\, s(\varepsilon)\, r(\varepsilon)^{1/2}}{c} \right) \overset{\varepsilon \downarrow 0}{\longrightarrow} 0\,,$$

where 0 is the null operator on the subspace $\mathcal{L}^\perp$.

With Theorem 3 at hand, the next task is to study the limit of the sequence of one-dimensional operators $(T_\varepsilon)_\varepsilon$. After a rather general result in Theorem 4, recovering the effective operators found in Theorems 1 and 2, we explicitly discuss two classes in the guise of illustrations, with norm resolvent convergence in case of vanishing bounded effective potentials $\varrho(x)$ at infinity and only strong convergence otherwise.

An additional hypothesis in the case of thick domains:

(iv) if $\lim_{x\to\infty} g(x) = \infty$, from now on we also suppose that $g(x)$ is strictly increasing for large enough x.

In addition, we assume that the number $c > 0$ in the definition of $t_\varepsilon(w)$ is so that $\varrho(x) + c > 0$, for all $x \in [a, \infty)$.

We perform an explicitly approximation of the Neumann Laplacian restricted to the thick region $\{(x, y) \in \mathbb{R}^2 \mid 0 < y < \varepsilon g(x),\ x \in [a, \infty)\}$, according to the lines already mentioned, that is, we will explicitly choose g_ε. Fix a number $0 < \alpha < 1$ and take $z_\varepsilon = z_\varepsilon(\alpha) > a$ so that $g(z_\varepsilon) = 1/\varepsilon^\alpha$; by *(iv)*, z_ε is uniquely determined for all $\varepsilon > 0$ small enough and $z_\varepsilon \to \infty$, as $\varepsilon \to 0$. Let

$$g_\varepsilon(x) := \begin{cases} g(x), & \text{if} \quad a \le x < z_\varepsilon, \\ 1/\varepsilon^\alpha, & \text{if} \quad\quad z_\varepsilon \le x. \end{cases}$$

Although continuous, the function g_ε is not differentiable at the point z_ε, and in calculations we may consider separately the regions $x \in [a, z_\varepsilon)$ and $x \in [z_\varepsilon, \infty)$.

Note that with such choice of g_ε, since $\alpha < 1$, the condition *(ii)* is automatically satisfied; if *(iii)* also holds, then Theorem 3 applies. Note that $\varepsilon g_\varepsilon$ vanishes uniformly as $\varepsilon \to 0$. Ahead, we give conditions so that the corresponding sequence $(T_\varepsilon)_\varepsilon$ converges in the resolvent sense to an operator independent of ε.

Recall the definition of the quadratic form $t_\varepsilon(w)$ and note that, in this situation,

$$\frac{g'_\varepsilon(x)}{2 g_\varepsilon(x)} = \chi_{[a, z_\varepsilon)}(x) j(x).$$

$\chi_I(x)$ denotes the characteristic function of the interval $I \subset \mathbb{R}$.

Let T be the self-adjoint operator

$$(Tw)(x) := -w''(x) + \varrho(x)w(x) + cw(x), \tag{4}$$

with $\operatorname{dom} T = \{w \in H^2([a,\infty)) \mid j(a)\, w(a) = w'(a)\}$. Theorem 4 characterizes (4) as an effective operator in this approximation process.

Theorem 4. *Let $g : [a,\infty) \to \mathbb{R}$ define a thick region and g_ε as above. Then:*

(*A*) *The sequence T_ε converges in the strong resolvent sense to the self-adjoint operator T. I.e., for each $f \in \mathrm{L}^2([a,\infty))$, $T_\varepsilon^{-1} f \to T^{-1} f$, as $\varepsilon \to 0$.*

(*B*) *Suppose that $j(x)$ vanishes as $x \to \infty$ and take $\zeta(\varepsilon)$ so that $\chi_{[z_\varepsilon,\infty)}(x) j(x) \le \zeta(\varepsilon)$, for all $x \in [a,\infty)$, with $\zeta(\varepsilon) \to 0$, as $\varepsilon \to 0$. Then, there exists $K > 0$, so that, for all $\varepsilon > 0$ small enough,*
$$\left\| T_\varepsilon^{-1} - T^{-1} \right\| \le K\, \zeta(\varepsilon).$$

By Theorems 3 and 4, in both cases (A) and (B), T is the effective operator we were looking for, and it coincides with S obtained from the collapsing of Λ_ε in Subsection 1.1 (for bounded or suitable unbounded g). Especially in case (B) we have norm convergence

$$\left\| H_\varepsilon^{-1} - \left(T^{-1} \oplus 0\right) \right\| \to 0, \qquad \varepsilon \to 0.$$

Proposition 1. *Let g be as above. Since $j(x)$ is a bounded function, then there exist $\gamma, \kappa > 0$ so that $g(x) \le \gamma\, e^{\kappa x}$, for all $x \in [a,\infty)$.*

Proof. By hypothesis, there is $\kappa > 0$ so that for all x one has $g'(x)/g(x) = 2j(x) \le \kappa$. Hence, there is $\tilde\gamma > 0$ so that

$$g(x) = \tilde\gamma\, \exp\left(\int_a^x 2j(s)\, \mathrm{d}s \right) \le \tilde\gamma\, e^{\kappa(x-a)} = \gamma\, e^{\kappa x},$$

with $\gamma = \tilde\gamma\, e^{-\kappa a}$. □

Since for $g(x) = \gamma\, e^{\kappa x}$ one has that $j(x) = \kappa/2$ is actually bounded (and $j'(x) = 0$), and combined with Proposition 1 it follows that the fastest growth permitted for $g(x)$ in order to apply Theorem 4 is the exponential one. In what follows, we discuss two classes of thick domains to exemplify the possibilities in Theorem 4 (also exemplifying Theorem 2).

Class I. [Power law] Let $g : [1,\infty) \to \mathbb{R}$ be the unbounded function $g(x) = \gamma\, x^\beta$, $\gamma, \beta > 0$ (so (i) and (iv) hold). Let $\alpha < 1$ and, for each $\varepsilon \in (0, 1/\gamma^{1/\alpha})$, let $z_\varepsilon = 1/(\gamma\varepsilon^\alpha)^{1/\beta}$; the above construction of g_ε gives, for this class,

$$g_\varepsilon(x) = \begin{cases} \gamma\, x^\beta, & \text{if } \; 1 \le x \le z_\varepsilon, \\ 1/\varepsilon^\alpha, & \text{if } \; z_\varepsilon < x. \end{cases}$$

Note that

$$g_\varepsilon(x) \le \frac{1}{\varepsilon^\alpha} \quad \text{and} \quad \left(\frac{g_\varepsilon'(x)}{g_\varepsilon(x)} \right)^2 \le \frac{\beta^2}{x^2} \le \beta^2, \qquad \forall x \in [1,\infty) \backslash \{z_\varepsilon\}.$$

Take $s(\varepsilon) = 1/\varepsilon^\alpha$ and $r(\varepsilon) = \beta^2$. Since $\alpha < 1$, conditions (ii) and (iii) are also satisfied.

Denote by A_ε the self-adjoint operator associated with the quadratic form

$$a_\varepsilon(w) := h_\varepsilon(w\, 1) = \int_1^\infty \left(\left| w' - V_\varepsilon(x) w \right|^2 + c|w|^2 \right) \mathrm{d}x, \quad \operatorname{dom} a_\varepsilon = H^1([1,\infty)),$$

where $V_\varepsilon(x) = \beta/(2x)\,\chi_{[1,z_\varepsilon)}(x)$. Note that $a_\varepsilon(w)$ is just the specialization to this case of the quadratic form $t_\varepsilon(w)$ in (3). The change of notation is to avoid confusion with Class II ahead.

As a consequence of Theorem 3, we have

Corollary 1. *There exists $K > 0$, so that, for all $\varepsilon > 0$ small enough,*

$$\left\| H_\varepsilon^{-1} - \left(A_\varepsilon^{-1} \oplus 0\right)\right\| \le K\,\varepsilon^{1-\alpha},$$

where 0 is the null operator on the subspace $\mathcal{L}^\perp$.

Next we analyze the convergence of the sequence A_ε. In this Class one has $a = 1, j(x) = \beta/(2x)$, $\varrho(x) = \frac{\beta(\beta-2)}{4x^2}$, and one can check that

$$\chi_{(z_\varepsilon,\infty)}(x) j(x) \le \frac{\beta}{2}\, z_\varepsilon^{-1}, \qquad \forall x \in [1,\infty),$$

with $z_\varepsilon^{-1} \to 0$, as $\varepsilon \to 0$. By item (B) in Theorem 4, we have

Corollary 2. *There exists $K > 0$, so that, for all $\varepsilon > 0$ small enough,*

$$\left\| A_\varepsilon^{-1} - A^{-1}\right\| \le K\, z_\varepsilon^{-1} \sim \varepsilon^{\alpha/\beta},$$

with

$$(Aw)(x) = -w''(x) + \frac{\beta(\beta-2)}{4x^2} w(x) + cw(x)$$

and $\operatorname{dom} A := \left\{ w \in H^2([1,\infty)) \mid (\beta/2)\, w(1) = w'(1) \right\}$.

By combining Corollaries 1 and 2, we conclude that

$$\left\| H_\varepsilon^{-1} - \left(A^{-1} \oplus 0\right)\right\| \sim K(\varepsilon^{1-\alpha} + \varepsilon^{\alpha/\beta}) \longrightarrow 0, \quad \varepsilon \to 0,$$

so that A is the effective operator in this situation.

The effective potential $\varrho(x)$ in this case:

- does not depend on γ;
- vanishes for $\beta = 2$ and is proportional to x^{-2} for all values of β;
- is attractive for $0 < \beta < 2$ and repulsive for $\beta > 2$.

Class II. [Exponential of a power] In this Class we consider $g : [1,\infty) \to \mathbb{R}$, $g(x) = \gamma\, e^{x^\beta}$, $\gamma > 0, 0 < \beta \le 1$ (similarly for $g(x) = \gamma\, e^{\kappa x^\beta}$, $\kappa > 0$), so (i) and (iv) hold. For each $0 < \alpha < 1$ and $\varepsilon \in (0, 1/(\gamma\, e)^{1/\alpha})$, pick $z_\varepsilon = (\ln(1/\gamma\,\varepsilon^\alpha))^{1/\beta}$ so that $g_\varepsilon : [1,\infty) \to \mathbb{R}$ is the function

$$g_\varepsilon(x) = \begin{cases} \gamma\, e^{x^\beta}, & \text{if } \; 1 \le x < z_\varepsilon, \\ 1/\varepsilon^\alpha, & \text{if } \; z_\varepsilon \le x. \end{cases}$$

Note that

$$g_\varepsilon(x) \le \frac{1}{\varepsilon^\alpha} \quad \text{and} \quad \left(\frac{g_\varepsilon'(x)}{g_\varepsilon(x)}\right)^2 \le \beta^2\, z_\varepsilon^{2(\beta-1)}, \qquad \forall x \in [1,\infty)\backslash\{z_\varepsilon\}.$$

In this case, we take $s(\varepsilon) = 1/\varepsilon^\alpha$ and $r(\varepsilon) = \beta^2\, z_\varepsilon^{2(\beta-1)}$, and conditions (ii)-(iii) are also satisfied.

Let B_ε be the self-adjoint operator associated with the quadratic form

$$b_\varepsilon(w) := h_\varepsilon(w\,1) = \int_1^\infty \left(\left|w' - Z_\varepsilon(x)w\right|^2 + c|w|^2\right) \mathrm{d}x, \quad \operatorname{dom} b_\varepsilon = H^1([1,\infty))\,,$$

with $Z_\varepsilon(x) = (\beta x^{\beta-1}/2)\,\chi_{[1,z_\varepsilon)}(x)$. Note that $b_\varepsilon(w)$ is just $t_\varepsilon(w)$ in (3) in this particular case. As a consequence of Theorem 3, we have

Corollary 3. *There exists $K > 0$, so that, for all $\varepsilon > 0$ small enough,*

$$\left\|H_\varepsilon^{-1} - \left(B_\varepsilon^{-1} \oplus 0\right)\right\| \le I(\varepsilon),$$

where $I(\varepsilon) := \varepsilon^{2(1-\alpha)}/\pi^2 + K\,\varepsilon^{1-\alpha}\,z_\varepsilon^{\beta-1}/c$, and 0 is the null operator on the subspace $\mathcal{L}^\perp$. Clearly $I(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Consider the self-adjoint operator

$$(Bw)(x) = (B^\beta w)(x) := -w''(x) + \varrho^\beta(x)w(x),$$

with potential $\varrho^\beta(x) := (1/4)\left(2\beta(\beta-1)x^{\beta-2} + \beta^2 x^{2(\beta-1)}\right) + c$ and domain

$$\operatorname{dom} B := \left\{w \in H^2([1,\infty)) \mid (\beta/2)w(1) = w'(1)\right\}.$$

If $0 < \beta < 1$, one has

$$\chi_{(z_\varepsilon,\infty)}(x)j(x) \le (\beta/2)\,z_\varepsilon^{\beta-1}, \qquad \forall x \in [1,\infty),$$

with $z_\varepsilon^{\beta-1} \to 0$, as $\varepsilon \to 0$. Theorem 4 ensures that

Corollary 4. *(j) ($0 < \beta < 1$) There exists $K > 0$, so that, for all $\varepsilon > 0$ small enough,*

$$\left\|B_\varepsilon^{-1} - B^{-1}\right\| \le K\,z_\varepsilon^{\beta-1}.$$

(jj) ($\beta = 1$; the exponential function) The sequence B_ε converges in the strong resolvent sense to the self-adjoint operator B. I.e., for each $f \in \mathrm{L}^2([1,\infty))$, $B_\varepsilon^{-1}f \to B^{-1}f$ as $\varepsilon \to 0$.

By combining Corollaries 3 and 4, we conclude, for $0 < \beta < 1$, a norm resolvent convergence

$$\left\|H_\varepsilon^{-1} - \left(B^{-1} \oplus 0\right)\right\| \longrightarrow 0, \qquad \varepsilon \to 0,$$

and strong resolvent convergence for $g(x) = \gamma\,e^x$.

With respect to the action of B, we have some considerations about the effective potential $\varrho^\beta(x)$:

- for $0 < \beta < 1$, it tends to zero as $x \to \infty$. Furthermore, it is negative in a neighborhood of 1 and positive for large values of x;
- for $\beta = 1$, it is constant and equals to $1/4$;
- by considering a general exponential function $g(x) = e^{\kappa x}$, $\kappa > 0$, the effective potential is also constant and given by $\kappa^2/4$, which vanishes as $\kappa \to 0$, approaching the border where norm convergence holds. We have then a kind of sharp transition here at exponential functions.

Remark 1. Other explicitly possible choice of thick region is given by the "slowly diverging" $g : [2, +\infty) \to \mathbb{R}$, $g(x) = \ln(x)$. Performing approximations following the same ideas in the discussions of Classes I and II, there is a convergence in the norm resolvent sense to the effective operator, which has the form (4) (properly adapted) with potential $\varrho(x) = -(1 + 2\ln x)/(4x^2 \ln^2 x)$. Rather surprisingly, it still has a x^{-2} decaying for large values of x.

The rest of this work concerns the proofs of the results related to the recovery of effective operators for thick regions from approximations by uniformly collapsing ones, i.e., Theorems 3 and 4. By considering the Laplacian in Q_ε and its respective quadratic form, in Section 2 we carry out the necessary change of variables to properly work with the form presented in (2). In Section 3 we discuss the restriction of the problem to the subspace $\mathcal{L}$ and prove Theorem 3. Theorem 4 is proven in Section 4. The Appendix presents details of the limit quadratic forms and some operators with Robin boundary conditions.

2 Quadratic forms and change of variables

As in the Introduction, let $\varepsilon_0 > 0$ and, for each $\varepsilon \in (0, \varepsilon_0)$, $g_\varepsilon : [a, \infty) \to \mathbb{R}$ a bounded and positive function. For each $\varepsilon \in (0, \varepsilon_0)$, consider the region

$$Q_\varepsilon = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < \varepsilon\, g_\varepsilon(x), x \in [a, \infty)\},$$

and the Neumann Laplacian restricted to Q_ε. Our study is aimed at analyzing the sequence of quadratic forms

$$d_\varepsilon(\psi) = \int_{Q_\varepsilon} |\nabla \psi|^2 \,\mathrm{d}x\mathrm{d}y, \quad \mathrm{dom}\ d_\varepsilon = H^1(Q_\varepsilon).$$

As already mentioned, it is convenient to study the sequence

$$f_\varepsilon(\psi) = d_\varepsilon(\psi) + c \int_{Q_\varepsilon} |\psi|^2 \,\mathrm{d}x\mathrm{d}y, \quad \mathrm{dom}\ f_\varepsilon = \mathrm{dom}\ d_\varepsilon, \tag{5}$$

for some $c > 0$ that will be fixed ahead.

Next we perform some standard changes of variables in the sequence of quadratic forms (5). The first one is given by the map

$$\begin{array}{cccc} F_\varepsilon : & [a, \infty) \times (0, 1) & \to & Q_\varepsilon \\ & (x, y) & \mapsto & (x, \varepsilon\, y\, g_\varepsilon(x)) \end{array}.$$

The Jacobian matrix of F_ε is

$$M = \begin{pmatrix} 1 & \varepsilon\, y\, g'_\varepsilon(x) \\ 0 & \varepsilon\, g_\varepsilon(x) \end{pmatrix},$$

and $\det M = \varepsilon\, g_\varepsilon(x) > 0$, for all $x \in [a, \infty)$. Thus, F_ε defines a global diffeomorphism.

Let $Q = [a, \infty) \times (0, 1)$. Introducing the notation

$$\|\psi\|_\varepsilon^2 := \int_Q |\psi(x, y)|^2 \varepsilon\, g_\varepsilon(x) \,\mathrm{d}x\mathrm{d}y,$$

and the unitary operator

$$U_\varepsilon : \begin{array}{ccc} \mathrm{L}^2(Q_\varepsilon) & \to & \mathrm{L}^2(Q, \varepsilon g_\varepsilon(x)\,\mathrm{d}x\mathrm{d}y) \\ \psi & \mapsto & \psi \circ F_\varepsilon \end{array},$$

we obtain the sequence of quadratic forms

$$\tilde{h}_\varepsilon(\psi) = \|M^{-1}\nabla(U_\varepsilon\psi)\|_\varepsilon^2 + c\,\|U_\varepsilon\psi\|_\varepsilon^2, \quad \mathrm{dom}\;\tilde{h}_\varepsilon = H^1(Q).$$

However, we still denote $U_\varepsilon\psi$ by ψ. Explicitly,

$$\tilde{h}_\varepsilon(\psi) = \int_Q \varepsilon\left(\frac{1}{g_\varepsilon(x)}\left|g_\varepsilon(x)\,\psi' - y\,g_\varepsilon'(x)\,\psi_y\right|^2 + \frac{|\psi_y|^2}{\varepsilon^2 g_\varepsilon(x)} + c\,g_\varepsilon(x)\,|\psi|^2\right)\mathrm{d}x\mathrm{d}y\,.$$

Note that $\tilde{h}_\varepsilon(\psi)$ is acting in the Hilbert space $\mathrm{L}^2(Q, \varepsilon g_\varepsilon(x)\,\mathrm{d}x\mathrm{d}y)$, and the region of integration Q is independent of the parameter $\varepsilon > 0$. It is then opportune to consider the unitary operator

$$V_\varepsilon : \begin{array}{ccc} \mathrm{L}^2(Q) & \to & \mathrm{L}^2(Q, \varepsilon g_\varepsilon(x)\,\mathrm{d}x\mathrm{d}y) \\ \psi & \mapsto & \psi/(\varepsilon g_\varepsilon(x))^{1/2} \end{array},$$

in order to work in $\mathrm{L}^2(Q)$ with the usual measure of subsets of the plane. Performing this transformation, the quadratic form $\tilde{h}_\varepsilon(\psi)$ becomes (2) in the Introduction, that is,

$$h_\varepsilon(\psi) = \int_Q \left(\left|\psi' - \frac{g_\varepsilon'(x)}{2g_\varepsilon(x)}\psi - y\frac{g_\varepsilon'(x)}{g_\varepsilon(x)}\psi_y\right|^2 + \frac{|\psi_y|^2}{\varepsilon^2 g_\varepsilon^2(x)} + c\,|\psi|^2\right)\mathrm{d}x\mathrm{d}y\,,$$

with $\mathrm{dom}\;h_\varepsilon = H^1(Q)$, a subspace of $\mathrm{L}^2(Q)$. Recall that H_ε denotes the self-adjoint operator associated with $h_\varepsilon(\psi)$.

3 Proof of Theorem 3

As before, consider the closed subspace $\mathcal{L} = \{w(x)\,1 \mid w \in \mathrm{L}^2([a,\infty))\}$, and perform the decomposition

$$\mathrm{L}^2(Q) = \mathcal{L} \oplus \mathcal{L}^\perp. \tag{6}$$

Thus, each $\psi \in \mathrm{L}^2(Q)$ can be uniquely written as

$$\psi(x,y) = w(x)\,1 + \eta(x,y), \quad w \in \mathrm{L}^2([a,\infty)), \quad \eta \in \mathcal{L}^\perp.$$

Note that $w \in H^1([a,\infty))$ if $\psi \in \mathrm{dom}\;h_\varepsilon$. If $\eta \in \mathrm{dom}\;h_\varepsilon \cap \mathcal{L}^\perp$, then

$$\int_a^\infty \eta(x,y)\,\mathrm{d}y = 0 \quad \text{and} \quad \int_a^\infty \eta'(x,y)\,\mathrm{d}y = 0, \qquad \text{a.e.}[x]. \tag{7}$$

Further, by the min-max principle,

$$\int_Q |\eta_y|^2\,\mathrm{d}x\mathrm{d}y \geq \pi^2 \int_Q |\eta|^2\,\mathrm{d}x\mathrm{d}y, \tag{8}$$

where π^2 is the second eigenvalue of the Neumann Laplacian in $(0,1)$.

Denote by $h_\varepsilon(\psi_1,\psi_2)$ the sesquilinear form associated with the quadratic form $h_\varepsilon(\psi)$. Due to the decomposition (6), for $\psi \in \mathrm{dom}\;h_\varepsilon$ one has

$$\psi(x,y) = w(x)\,1 + \eta(x,y), \quad w \in H^1([a,\infty)), \quad \eta \in \mathrm{dom}\;h_\varepsilon \cap \mathcal{L}^\perp.$$

Recall that $t_\varepsilon(w)$ was defined in (3) in the Introduction. For simplicity, write $w(x)\,1 = w(x)$; thus $h_\varepsilon(\psi)$ can be rewritten as

$$h_\varepsilon(\psi) = t_\varepsilon(w) + h_\varepsilon(w,\eta) + h_\varepsilon(\eta,w) + h_\varepsilon(\eta)\,.$$

We are going to check that there are $c_0 > 0$ and functions $0 \le q(\varepsilon), 0 \le p(\varepsilon)$ and $c(\varepsilon)$ so that $t_\varepsilon(w)$, $h_\varepsilon(\eta)$ and $h_\varepsilon(w,\eta)$ satisfy the following conditions:

$$t_\varepsilon(w) \ge c(\varepsilon)\|w\|^2_{\mathrm{L}^2(Q)}, \quad \forall w \in \mathcal{H}^1[a,\infty), \quad c(\varepsilon) \ge c_0 > 0; \tag{9}$$

$$h_\varepsilon(\eta) \ge p(\varepsilon)\|\eta\|^2_{\mathrm{L}^2(Q)}, \quad \forall \eta \in \mathrm{dom}\ h_\varepsilon \cap \mathcal{L}^\perp; \tag{10}$$

$$|h_\varepsilon(w,\eta)|^2 \le q(\varepsilon)^2\, t_\varepsilon(w)\, h_\varepsilon(\eta), \quad \forall \psi \in \mathrm{dom}\ h_\varepsilon; \tag{11}$$

and with

$$p(\varepsilon) \to \infty, \quad c(\varepsilon) = O(p(\varepsilon)), \quad q(\varepsilon) \to 0 \quad \text{as} \quad \varepsilon \to 0. \tag{12}$$

Thus, Proposition 3.1 in [8] guarantees that, for $\varepsilon > 0$ small enough,

$$\left\| H_\varepsilon^{-1} - \left(T_\varepsilon^{-1} \oplus 0\right)\right\|_{\mathrm{L}^2(Q)} \le p(\varepsilon)^{-1} + K\, q(\varepsilon)\, c(\varepsilon)^{-1},$$

for some $K > 0$. Actually, this proposition was originally proven for real Hilbert spaces, but, with simple adaptations, it can be extended to complex Hilbert spaces.

We see that condition (9) is satisfied by noting that $t_\varepsilon(w) \ge c\int_a^\infty |w|^2 \mathrm{d}x$ and taking $c(\varepsilon) = c = c_0$. Condition (10) follows by (8) and hypothesis (ii). In fact,

$$h_\varepsilon(\eta) \ge \int_Q \frac{|\eta_y|^2}{\varepsilon^2 g_\varepsilon^2(x)}\,\mathrm{d}x\mathrm{d}y \ge \frac{1}{\varepsilon^2 s^2(\varepsilon)}\int_Q |\eta_y|^2\,\mathrm{d}x\mathrm{d}y \ge \frac{\pi^2}{\varepsilon^2 s^2(\varepsilon)}\int_Q |\eta|^2\,\mathrm{d}x\mathrm{d}y,$$

and just take $p(\varepsilon) = \pi^2/(\varepsilon s(\varepsilon))^2$.

It remains to analyze

$$h_\varepsilon(w,\eta) = \int_Q \left(w' - \frac{g_\varepsilon'(x)}{g_\varepsilon(x)}\frac{w}{2}\right)\left(\overline{\eta}' - \frac{g_\varepsilon'(x)}{g_\varepsilon(x)}\frac{\overline{\eta}}{2} - y\,\frac{g_\varepsilon'(x)}{g_\varepsilon(x)}\overline{\eta}_y\right)\mathrm{d}x\mathrm{d}y.$$

Due to (7), this expression reduces to

$$h_\varepsilon(w,\eta) = -\int_Q \left[\left(w' - \frac{g_\varepsilon'(x)}{g_\varepsilon(x)}\frac{w}{2}\right) y\,\frac{g_\varepsilon'(x)}{g_\varepsilon(x)}\,\overline{\eta}_y\right]\mathrm{d}x\mathrm{d}y.$$

By (iii), we have

$$\begin{aligned} |h_\varepsilon(w,\eta)| &\le t_\varepsilon(w)^{1/2}\left(\int_Q \left(\frac{g_\varepsilon'(x)}{g_\varepsilon(x)}\right)^2 |\eta_y|^2\,\mathrm{d}x\mathrm{d}y\right)^{1/2} \\ &\le r(\varepsilon)^{1/2}\, t_\varepsilon(w)^{1/2}\left(\int_Q |\eta_y|^2\,\mathrm{d}x\mathrm{d}y\right)^{1/2} \\ &\le \varepsilon\, s(\varepsilon)\, r(\varepsilon)^{1/2}\, t_\varepsilon(w)^{1/2}\, h_\varepsilon(\eta)^{1/2}. \end{aligned}$$

We take $q(\varepsilon) = \varepsilon\, s(\varepsilon)\, r(\varepsilon)^{1/2}$ and so conditions (11) and (12) are satisfied as well. The proof of Theorem 3 is complete.

Remark 2. As a simple consequence of Theorem 3, suppose that $g : (a,b) \to \mathbb{R}$ is a bounded function of class C^2 satisfying

$$c_1 \le g(x) \le c_2, \quad \left|\frac{g'(x)}{g(x)}\right| \le c_3, \qquad \forall x \in (a,b),$$

for some numbers $c_1, c_2, c_3 > 0$. In this case, it is not necessary to perform an approximation g_ε as in the Introduction to get uniform convergence. We can work directly with the Neumann Laplacian restricted in the region $\{(x,y) \in \mathbb{R}^2 \mid 0 < y < \varepsilon\, g(x), x \in (a,b)\}$. The effective operator turns out to be the operator associated with the quadratic form

$$\int_a^b \left(\left| w' - \frac{g'(x)}{2g(x)} w \right|^2 + c|w|^2 \right) \mathrm{d}x, \qquad w \in H^1(a,b). \tag{13}$$

A simple adaptation of the proof of Theorem 3 shows that the same result is obtained if we take the sequence $(g(x,\varepsilon))_\varepsilon$ considered by Hale and Raugel [9] (under the conditions (2.1) in Section 2 of [9]). The limit quadratic form is again (13), but with $g(x)$ replaced by $g_0(x) = g(x,0)$.

4 Proof of Theorem 4

Recall that we denote by T_ε the self-adjoint operator associated with the quadratic form $t_\varepsilon(w)$ in (3) and $\varrho(x) = j^2(x) + j'(x)$. By general arguments (e.g., see [4], p. 101), one has,

$$\begin{aligned} \mathrm{dom}\, T_\varepsilon \;=\; & \big\{ w \in \mathrm{dom}\, t_\varepsilon \mid \text{there exists } \varphi \in L^2([a,\infty)) \text{ with } t_\varepsilon(\psi, w) = \langle \psi, \varphi \rangle, \\ & \forall \psi \in \mathrm{dom}\, t_\varepsilon \big\}, \end{aligned}$$

and $T_\varepsilon w = \varphi$. Here, $t_\varepsilon(\psi, w)$ denotes the sesquilinear form corresponding to $t_\varepsilon(w)$.

Let

$$\mathcal{C} := \big\{ w \in C_0^\infty[a,\infty) \mid j(a)w(a) = w'(a) \big\}$$

and denote $j_\varepsilon := g_\varepsilon'/(2g_\varepsilon)$. Given $w \in \mathcal{C}$, for all small $\varepsilon > 0$ so that support$(w) \subset [a, z_\varepsilon)$, one has $j_\varepsilon(x) = j(x)$ for $x \in [a, z_\varepsilon)$ and a direct evaluation shows that

$$t_\varepsilon(\psi, w) = \langle \psi, -w'' + \varrho(x) w + cw \rangle, \qquad \forall \psi \in \mathrm{dom}\, t_\varepsilon.$$

Thus, such $w \in \mathrm{dom}\, T_\varepsilon$ and

$$(T_\varepsilon w)(x) = -w''(x) + \varrho(x) w(x) + c w(x),$$

for all $\varepsilon > 0$ small enough.

Due to the hypotheses on g and j, it follows by the results discussed in the Appendix that $\mathcal{C}$ is a core of T and the quadratic form $t(w)$ associated with the operator T is

$$\begin{aligned} t(w) &= \int_a^\infty \left(|w'|^2 + (\varrho(x) + c)|w|^2 \right) \mathrm{d}x + j(a)|w(a)|^2 \\ &= \int_a^\infty \left(\left| w' - j(x) w \right|^2 + c|w|^2 \right) \mathrm{d}x\,, \end{aligned}$$

dom $t = H^1([a,\infty))$.

Proof of (A). Since $\mathcal{C}$ is a core of T, $\mathcal{D} := T(\mathcal{C})$ is a dense set in $\mathrm{L}^2([a,\infty))$. Also note that for any $0 < c_4 < c$ one has $t_\varepsilon > c_4$. Thus, uniformly in (small) $\varepsilon > 0$,

$$\sup\left\{\|T_\varepsilon^{-1}\|\right\} \le 1/c_4. \tag{14}$$

For each $w \in \mathcal{C}$, denote $f = Tw \in \mathcal{D}$. By the second resolvent identity, one finds

$$\begin{aligned}\left\|\left(T_\varepsilon^{-1} - T^{-1}\right) f\right\| &= \|T_\varepsilon^{-1}(T_\varepsilon - T)w\| \\ &\le (1/c_4)\|(T_\varepsilon - T)w\| \longrightarrow 0, \quad \varepsilon \to 0,\end{aligned}$$

since $T_\varepsilon w = Tw$, for all $\varepsilon > 0$ small enough.

By (14), the convergence in the dense set $\mathcal{D}$ can be extended to $\mathrm{L}^2([a,\infty))$, that is,

$$\left\|\left(T_\varepsilon^{-1} - T^{-1}\right) f\right\| \longrightarrow 0, \quad \varepsilon \to 0,$$

for all $f \in \mathrm{L}^2([a,\infty))$.

Proof of (B). In this case one has $\mathrm{dom}\, t = \mathrm{dom}\, t_\varepsilon = H^1([a,\infty))$, for all $\varepsilon > 0$. Let $Y_\varepsilon(x) := \chi_{[a,z_\varepsilon)}(x) j(x)$ and $\tilde{Y}_\varepsilon(x) := \chi_{[z_\varepsilon,\infty)}(x) j(x)$. By hypothesis,

$$\tilde{Y}_\varepsilon(x) \le \zeta(\varepsilon), \qquad \forall x \in [a,\infty).$$

There exists $K_1 > 0$, so that, for all $w \in H^1([a,\infty))$,

$$\begin{aligned} t_\varepsilon(w) &= \int_a^\infty \left(\left|w' - Y_\varepsilon(x)w - \tilde{Y}_\varepsilon(x)w + \tilde{Y}_\varepsilon(x)w\right|^2 + c|w|^2\right) \mathrm{d}x \\ &\le \left[\left(\int_a^\infty \left|w' - j(x)w\right|^2 \mathrm{d}x\right)^{1/2} + \left(\int_a^\infty \tilde{Y}_\varepsilon^2(x)|w|^2 \mathrm{d}x\right)^{1/2}\right]^2 + c\int_a^\infty |w|^2\,\mathrm{d}x \\ &\le t(w) + 2\,t(w)^{1/2}\left(\int_a^\infty \tilde{Y}_\varepsilon^2(x)|w|^2\mathrm{d}x\right)^{1/2} + \int_a^\infty \tilde{Y}_\varepsilon^2(x)|w|^2\mathrm{d}x \\ &\le t(w) + 2\,\zeta(\varepsilon)\,t(w)^{1/2}\left(\int_a^\infty |w|^2\mathrm{d}x\right)^{1/2} + \zeta^2(\varepsilon)\int_a^\infty |w|^2\mathrm{d}x \\ &\le t(w) + K_1\,\zeta(\varepsilon)\,t(w). \end{aligned}$$

Thus,

$$t_\varepsilon(w) - t(w) \le K_1\,\zeta(\varepsilon)\,t(w), \qquad \forall w \in H^1([a,\infty)).$$

In particular, for all $\varepsilon > 0$ small enough,

$$t_\varepsilon(w) \le K_2\,t(w), \qquad \forall w \in H^1([a,\infty)), \tag{15}$$

for some $K_2 > 0$.

A similar calculation shows that

$$t(w) - t_\varepsilon(w) \le K_3\,\zeta(\varepsilon)\,t(w)^{1/2}\,t_\varepsilon(w)^{1/2}, \qquad \forall w \in H^1([a,\infty)),$$

for some $K_3 > 0$. Combining with (15), one has, for all $\varepsilon > 0$ small enough,

$$t(w) - t_\varepsilon(w) \le K_3\sqrt{K_2}\,\zeta(\varepsilon)\,t(w), \quad \forall w \in H^1([a,\infty)).$$

The inequalities above imply that, for all $\varepsilon > 0$ small enough,

$$|t_\varepsilon(w) - t(w)| \le \max\{K_1, K_3\sqrt{K_2}\}\zeta(\varepsilon)\,t(w), \qquad \forall w \in H^1([a,\infty)).$$

The result follows by Theorem 3 (Section 2.1) in [2].

A Appendix

A.1 Self-adjointness

We consider continuous potentials $V : [a, +\infty) \to \mathbb{R}$ in this appendix. Given $r \in \mathbb{R}$, let $\tau = \tau_r$ be the "minimal Robin operator"

$$(\tau w)(x) = -w''(x) + V(x)w(x),$$

with domain

$$\mathrm{dom}\ \tau = \left\{ w \in \mathrm{C}_0^\infty[a,\infty) \mid r\, w(a) = w'(a) \right\}.$$

In the sequel we discuss the essential self-adjointness of τ. The uniqueness of its self-adjoint extension can certainly be obtained from results in the literature, but we include this discussion since such characteristic is strongly used in this work.

Let $\mathcal{AC}[a,\infty)$ denote the set of absolutely continuous functions in each compact subinterval of $[a,\infty)$.

Theorem A.1. *The operator τ is Hermitian and its adjoint τ^* is*

$$(\tilde{\tau} w)(x) = -w''(x) + V(x)w(x),$$

with

$$\begin{aligned} \mathrm{dom}\ \tilde{\tau} \ = \ & \left\{ w \in \mathrm{L}^2([a,\infty)) \mid w, w' \in \mathcal{AC}[a,\infty), r\, w(a) = w'(a) \textit{ and} \right. \\ & \left. \tilde{\tau} w \in \mathrm{L}^2([a,\infty)) \right\}. \end{aligned}$$

Proof. It is immediate to check that τ is Hermitian. Some simple calculations show that $\mathrm{dom}\ \tilde{\tau} \subset \mathrm{dom}\ \tau^*$. Next we argue that $\mathrm{dom}\ \tau^* \subset \mathrm{dom}\ \tilde{\tau}$. By definition, if $u \in \mathrm{dom}\ \tau^*$, then

$$\langle \tau\phi, u \rangle = \langle \phi, \tau^* u \rangle, \qquad \forall \phi \in \mathrm{dom}\ \tau. \tag{16}$$

Since $u \in \mathrm{L}^2([a,\infty))$ and $V \in \mathrm{L}^2_{\mathrm{loc}}[a,\infty)$, we have $V(x)u \in \mathrm{L}^1_{\mathrm{loc}}[a,\infty)$. Further, $\tau^* u \in \mathrm{L}^1_{\mathrm{loc}}[a,\infty)$. Let $W(a)$ be a real number. We define

$$W(x) := r\ (x-a)\, W(a) + W(a) + \int_a^x \left(\int_a^s (V(t)u(t) - (\tau^* u)(t))\, \mathrm{d}t \right) \mathrm{d}s.$$

Since $(V(t)u - \tau^* u) \in \mathrm{L}^1_{\mathrm{loc}}[a,\infty)$, it follows that $W, W' \in \mathcal{AC}[a,\infty)$; we also have

$$W'(a) = r\ W(a), \quad W''(x) = V(x)u(x) - (\tau^* u)(x), \qquad \text{a.e.}[x]. \tag{17}$$

By (16) and (17), some calculations show that

$$\langle \phi'', u \rangle = \langle \phi'', W \rangle, \qquad \forall \phi \in \mathrm{dom}\ \tau.$$

This equation implies

$$W(x) = u(x), \qquad \text{a.e.}[x],$$

so that $u, u' \in \mathcal{AC}[a,\infty)$.

Upon integration by parts, (16) and (17) show that

$$\langle \tau\phi, u \rangle = \phi(a)[r\, \overline{u}(a) - \overline{u}'(a)] + \langle \phi, \tau^* u \rangle, \quad \forall \phi \in \mathrm{dom}\ \tau.$$

Since there exist functions $\phi \in \mathrm{dom}\ \tau$ so that $\phi(a) \neq 0$, we have $u'(a) = r\, u(a)$. Thus, $u \in \mathrm{dom}\ \tilde{\tau}$. □

Next, we shall make use of the limit point criterion (see [4, 12] for details).

Proposition A.1. *τ is in limit point case at a. If $V(x)$ is bounded from below, then τ is in the limit point case at ∞.*

Proof. Due the condition $u'(a) = r\,u(a)$ on $u \in \mathrm{dom}\ \tau^*$, τ is in limit point case at a. On the other hand, since $V(x)$ is bounded from below, Theorem 6.3 in [12] ensures that τ is in the limit point case at ∞. □

Corollary A.1. *If $V(x)$ is bounded from below, then τ is essentially self-adjoint (i.e., $\mathrm{dom}\ \tau$ is a core of τ^*). Hence, τ^* is its unique self-adjoint extension.*

Note that in case $V(x)$ is a bounded (continuous) function, then the domain of the unique self-adjoint extension above reduces to

$$\mathrm{dom}\ \tau^* = \left\{ w \in H^2([a,\infty)) \mid r\,w(a) = w'(a) \right\} ,$$

e.g., as for (4) in the Introduction.

A.2 Quadratic form

In this subsection we stick it out with $g : [a, +\infty) \to \mathbb{R}$ of class C^2 and bounded continuous functions $j(x), j'(x)$, so bounded potential $\varrho(x) = j^2(x) + j'(x)$. The usual quadratic form associated with the self-adjoint operator T in (4) is given by

$$t(w) = \int_a^\infty \left[|w'|^2 + (\varrho(x) + c)|w|^2 \right] \mathrm{d}x + j(a)|w(a)|^2,$$

with $\mathrm{dom}\ t = H^1([a,\infty))$.

Pick $w \in \mathrm{dom}\ t$. Since $j(x)|w(x)|^2 \to 0$ as $x \to \infty$, an integration by parts implies

$$\begin{aligned} t(w) &= \int_a^\infty \left[|w'|^2 + (j^2(x) + j'(x) + c)|w|^2 \right] \mathrm{d}x + j(a)|w(a)|^2 - \lim_{x\to\infty} j(x)|w(x)|^2 \\ &= \int_a^\infty \left[|w'|^2 + j^2(x)|w|^2 - j(x)(\bar{w}\,w' + \bar{w}'w) + c|w|^2 \right] \mathrm{d}x \\ &= \int_a^\infty \left(|w' - j(x)w|^2 + c|w|^2 \right) \mathrm{d}x, \end{aligned}$$

and we have obtained another expression for t, one that naturally appears in the calculations in the main part of this text.

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