Higher Order Endomorphisms

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A higher order endomorphism is a type of discrete dynamical system with memory. This note is an announcement of the main results of our doctoral thesis presented at IMPA in 1980. I am grateful to my advisor professor Jorge Sotomayor and to CNPq for financial support during the preparation of this work.

1. INTRODUCTION

Let $M$ be a smooth compact boundaryless manifold. Let $k$ be an integer $\geq 1$. We denote by $M^k$ the product manifold $M \times \ldots \times M$ ($k$-times).

1.1. Definition. An endomorphism of order $k$ (or $k$-endomorphism) is a map $f: M^k \to M$. A sequence $(x_n)_{n \geq 1}$, $x_n \in M$ for all $n \geq 1$, is an orbit of $f$ if it satisfies the condition $f(x_{n+k}) = x_{n+k}$ for all $n \geq 1$. Hence, the orbit $(x_n)_{n \geq 1}$ is determined by the $k$-tuple $(x_1, \ldots, x_k)$.

An endomorphism $f: M \to M$ is called a 1-endomorphism in our notation. If $k \geq 2$ we say that such a $k$-endomorphism is a higher order endomorphism.

In the classical literature objects like higher order endomorphisms were called recurrences. See (M).

1.2. We will describe the dynamics of a $k$-endomorphism $f: M^k \to M$ using the endomorphism $\tilde{f}$, defined by $\tilde{f}(x_1, \ldots, x_k) = (x_2, \ldots, x_k, f(x_1, \ldots, x_k))$, which we will call the lifting of $f$. There is a close relationship between the dynamics of $f$ and that of $\tilde{f}$. In fact, let $(x_n)_{n \geq 1}$ be an orbit of $f$. Then the orbit of $\tilde{f}$ beginning in $(x_1, \ldots, x_k)$ is $\{(x_n, x_{n+k-1}) : n \geq 1\}$. Moreover every orbit of $\tilde{f}$ has this form.

1.3. Definition. A periodic orbit of $f$ is a $p$-tuple $(x_1, \ldots, x_p)$ such that $f(x_1, \ldots, x_k) = x_{k+1}$, $f(x_2, \ldots, x_{k+1}) = x_{k+2}$, $\ldots$, $f(x_{p-k+2}, \ldots, x_p, x_1) = x_{p-k+2}$, $f(x_{p-k+2}, \ldots, x_p, x_{k-1}) = x_k$ for all $k < p$. If $k \geq p$ the definition is analogous. In particular a fixed point of $f$ is an element $x \in M$ such that $f(x) = x$. The period of the periodic orbit $(x_1, \ldots, x_p)$ is $p$ if it is minimal with respect to the above conditions.

1.4. The correspondence between orbits of $f$ and $\tilde{f}$ established in 1.2 preserves periodic orbits and their periods. At each periodic orbit $(x_1, \ldots, x_p)$ of $f$ with period $p$ corresponds the periodic orbit $\{(x_1, \ldots, x_k), (x_2, \ldots, x_{k+1}), \ldots, (x_{p-k+2}, \ldots, x_p, x_1, \ldots, x_{k-1})\}$ of $\tilde{f}$ with period $p$. Moreover, every periodic orbit of $\tilde{f}$ has this form. Furthermore this correspondence preserves invariant sets, $\omega$-limit sets and some kinds of attractors.

2. GENERIC PROPERTIES OF $k$-ENDOMORPHISMS

2.1. Definitions. Let $f \in C^r(M^k, M)$, $r \geq 1$, and let $(x_1, \ldots, x_p)$ be a periodic orbit of $f$ with period $p$. This orbit is hyperbolic if every eigenvalue of $Df(x_1, \ldots, x_k)$ has norm $\neq 0, 1$.

Let $H_p = \{f \in C^r(M^k, M) : \text{every periodic orbit of } f \text{ with period } \leq p \text{ is hyperbolic}\}$

and let \( KS'(M^k, M) \) be the set of \( f \in C'(M^k, M) \)
which satisfy:

a) The periodic orbits of \( f \) are all hyperbolic;
b) If \( (x_1, \ldots, x_n) \) is a periodic orbit of \( f \) with period \( p \), let \( W^s_{loc}(x_1, \ldots, x_n) \) be the local stable manifold of \( f \) at \( (x_1, \ldots, x_n) \).

Then the set \( W^s(x_1, \ldots, x_n) = \{ x \in M : \exists \ n \in Z, \ f^n(x) \in W^s_{loc}(x_1, \ldots, x_n) \} \) is a 1-1 immersed submanifold of constant dimension.

c) If \( (y_1, \ldots, y_n) \) is another periodic orbit of \( f \) with period \( q \) and \( W^u_{loc}(y_1, \ldots, y_n) \) is the local unstable manifold of \( f \) at \( (y_1, \ldots, y_n) \), then \( f^n | W^u_{loc}(y_1, \ldots, y_n) \) is transversal to \( W^s(x_1, \ldots, x_n) \).

We say that \( f \in KS'(M^k, M) \) is a Kupka-Smale \( k \)-endomorphism.

The following theorem extends a result of M. Shub. See (S).

2.2 Theorem. \( H_p \) is open and dense in \( C'(M^k, M) \) for all \( p \geq 1 \). Moreover \( KS'(M^k, M) \) is residual in \( C'(M^k, M) \).

2.3. Remark. Note that the map \( C'(M^k, M) \to C'(M^k, M), f \to \tilde{f} \), is an embedding and \( C'(M^k, M) \) is a submanifold of \( C'(M^k, M) \). Then theorem 2.2 says that \( KS'(M^k, M) \cap C'(M^k, M) \) is residual in \( C'(M^k, M) \), where \( KS'(M^k) \) is the space of \( C' \) Kupka-Smale endomorphisms of \( M^k \).

3. A STABILITY THEOREM

3.1. Let \( k \) and \( s \) be integers such that \( k \geq s + 1 \) and \( s \geq 1 \). Let \( C'_s(M^k, M) \) denote the subspace of \( C'(M^k, M) \) consisting of the elements \( f \in C'(M^k, M) \) which satisfy the following conditions:

a) \( f \) is injective in the first variable; and
b) \( D_x f(p) \) is an isomorphism for all \( p \in M^k \).

\( C'_s(M^k, M) \) is an open subspace of \( C'(M^k, M) \) and \( C'(M^k, M) \) is considered embedded in \( C'(M^k, M) \) by the map \( f \to f_0, f_0(x_1, \ldots, x_k) = x_i(x_{k-s+1}, \ldots, x_k) \).

Hence \( f_0 \in C'_s(M^k, M) \) is a \( k \)-endomorphism independent of the first \( k-s \) variables.

The theorem below asserts that every \( k \)-endomorphism sufficiently close to \( f_0 \) in the \( C' \) topology behaves like an \( s \)-endomorphism.

Before we examine the particular case \( k = 2 \) and \( s = 1 \).

3.2. Example. Let \( f \in C'_1(M^k, M) \) (hence \( f \) is a diffeomorphism) and let \( \gamma : M^k \to M, \gamma(x,y) = f(x) \).

The graph of \( \gamma \) is \( \gamma \)-invariant \( C' \) submanifold of \( M^2 \) and \( \tilde{\gamma} \) is a \( C' \) diffeomorphism. The "horizontal lines" \( \tilde{M}_p = \{(x, y) : x \in M \} \) where \( p = (\phi^{-1}(y), y) \in \gamma \) constitute a stable invariant family for \( \tilde{\gamma} \) such that \( \tilde{\gamma}(\tilde{M}_p) = \{(y, \phi(y)) : y \in \gamma \} \) for all \( p \in \gamma \). We have \( \tilde{\gamma}(M^2) = \gamma \) and \( \tilde{\gamma} | \gamma \) is conjugate to \( \phi \). This structure is persistent in \( C'(M^k, M) \) according to the following

3.3. Theorem. Let \( f \in C'_s(M^k, M) \) and \( f_0 : M^k \to M \) given by \( f_0(x_1, \ldots, x_k) = \phi(x_{k-s+1}, \ldots, x_k) \).

Let \( \gamma : M^k \to M_0 \) be the lifting of \( f_0 \). Then there exists a neighborhood \( \eta \) of \( f \) in \( C'(M^k, M) \) such that every \( g \in \eta \) has an invariant submanifold \( V_g \) diffeomorphic to \( M^k \), \( g | V_g \) is a \( C' \) diffeomorphism and \( V_g \subset V_{\tilde{\gamma}} \) if \( g \to \tilde{\gamma} \). Moreover there exists a neighborhood \( U \) of \( V_{\tilde{\gamma}} \) in \( M^k \) such that for all \( g \in \eta \) we have \( g^{-1}(M^k) \subset U \) and \( U \) is foliated by a continuous family \{\( W_{\gamma}(g, p) \)\} of \( C' \) stable discs pairwise disjoint and transversal to \( V_{\tilde{\gamma}} \).

Furthermore if \( g \in \eta \cap C'(M^k, M) \) then there exists a \( f_0 \in C'(M^k, M) \) whose lifting \( f : M^k \to M_0 \) is conjugate to \( \psi | V' \).

REFERENCES

